Laplace Transform Properties

There are a number of properties that can simplify taking Laplace Transforms and finding their inverse. We'll cover a few properties here and you can read about the rest in the textbook and in the Irwin Power Point Lecture notes for Chapters 13 (Laplace Transform) and 14 (Laplace Transform Applications) which cover the properties and their use in the puzzle solving that is involved in doing the Inverse Laplace Transform without resorting to doing a contour integral in the complex s-plane.

Real Time Shifting

$$x(t)u(t) \leftrightarrow X(s)$$
$$x(t-t_0)u(t-t_0) \leftrightarrow e^{-t_0 s} X(s)$$

Derive this:

Plugging in the time-shifted version of the function into the Laplace Transform definition, we get:

$$\int_{t=-\infty}^{\infty} x(t-t_0)u(t-t_0)e^{-st}dt$$
$$= \int_{t=t_0}^{\infty} x(t-t_0)e^{-st}dt$$

Letting $\tau = t - t_0$, we get:

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau$$
$$= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau$$
$$= e^{-st_0}X(s)$$

Differentiation

$$x(t) \leftrightarrow X(s)$$

$$x'(t) \leftrightarrow sX(s) - x(0^+)$$

↳ **for Unilateral Laplace Transform only**

Recall the equation for the voltage of an inductor:

$$V_L(t) = L\frac{di_L(t)}{dt}$$

If we take the Laplace Transform of both sides of this equation, we get:

$$V_L(s) = sLI_L(s)$$

which is consistent with the fact that an inductor has impedance *sL*.

Proof of the Differentiation Property:

1) First write $x(t)$ using the Inverse Laplace Transform formula:

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)\,e^{st}\,ds$$

2) Then take the derivative of both sides of the equation with respect to t (this brings down a factor of s in the second term due to the exponential):

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} sX(s)\,e^{st}\,ds$$

3) This shows that $x'(t)$ is the Inverse Laplace Transform of $s\,X(s)$:

$$\frac{d}{dt}x(t) \leftrightarrow sX(s)$$

The Differentiation Property is useful for solving differential equations.

Integration

$$x(t) \leftrightarrow X(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X(s)$$

Recall the equation for the voltage of a capacitor turned on at time 0:

$$V_c(t) = \frac{1}{C}\int_{0}^{t} i_c(\tau)\,d\tau$$

If we take the Laplace Transform of both sides of this equation, we get:

$$V_C(s) = \frac{1}{(sC)}I_C(s)$$

which is consistent with the fact that a capacitor has impedance $\frac{1}{sC}$.

Additional Properties

Multiplication by *t*

$$x(t) \leftrightarrow X(s)$$
$$t\,x(t) \leftrightarrow -\frac{dX(s)}{ds}$$

Derive this:

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

Take the derivative of both sides of this equation with respect to *s*:

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} x(t)(-t\,e^{-st})\,dt \; = \int_{-\infty}^{\infty} (-t\,x(t))\,e^{-st}\,dt$$

This is the expression for the Laplace Transform of -*t* *x*(*t*). Therefore,

$$t\,x(t) \leftrightarrow -\frac{dX(s)}{ds}$$

Initial Value

$$x(0^{+}) = \lim_{s \to \infty} sX(s)$$

(Given without proof)

Final Value

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

(Given without proof)

Independent-Variable Transformation (for Unilateral Laplace Transform)

$$x(t) \leftrightarrow X(s)$$
$$x(at-b) \leftrightarrow \frac{1}{a} e^{\frac{-sb}{a}} X\left(\frac{s}{a}\right)$$

Derive this:

Plugging in the definition, we find the Laplace Transform of $x(at\ -b)$:

$$\int_{-\infty}^{\infty} x(at-b)\, e^{-st} dt$$

Let $u = at - b$ and $du = adt$, we get:

$$= \int_{-\infty}^{\infty} x(u)\, e^{\frac{-s(u+b)}{a}} \frac{du}{a}$$
$$= \frac{1}{a} e^{\frac{-sb}{a}} \int_{-\infty}^{\infty} x(u)\, e^{\frac{-su}{a}} du$$
$$= \frac{1}{a} e^{\frac{-sb}{a}} X\left(\frac{s}{a}\right)$$