Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity

- Hann:

$$w[n] = 0.5 + 0.5\cos(\frac{2\pi n}{2M+1}), \quad -M \leq n \leq M$$

- Hamming:

$$w[n] = 0.54 + 0.46\cos(\frac{2\pi n}{2M+1}), \quad -M \leq n \leq M$$

- Blackman:

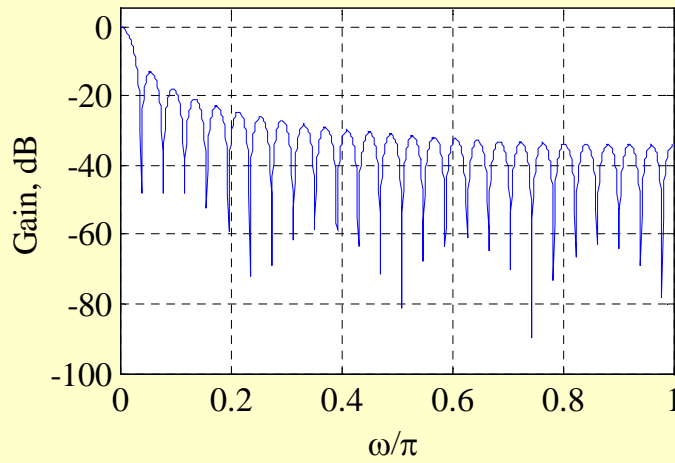$$w[n] = 0.42 + 0.5\cos(\frac{2\pi n}{2M+1}) + 0.08\cos(\frac{4\pi n}{2M+1})$$
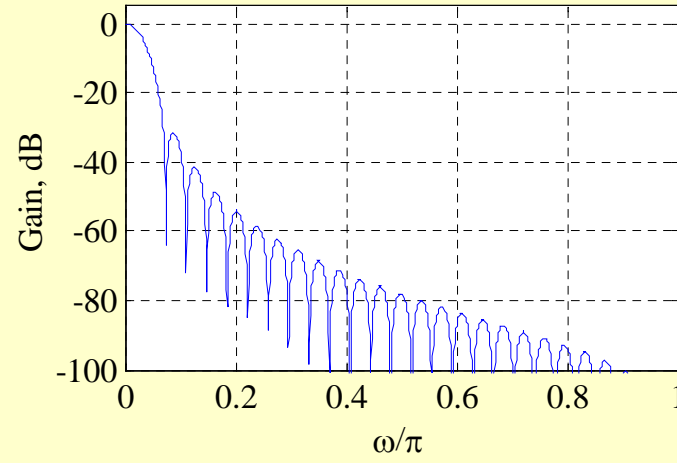
1

Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M = 25$ are shown below:
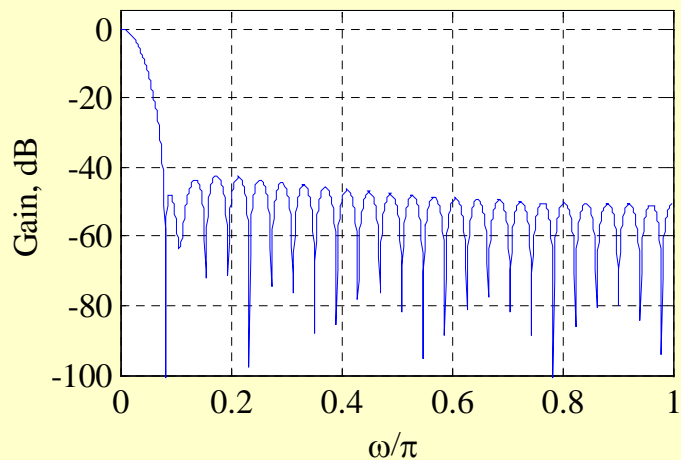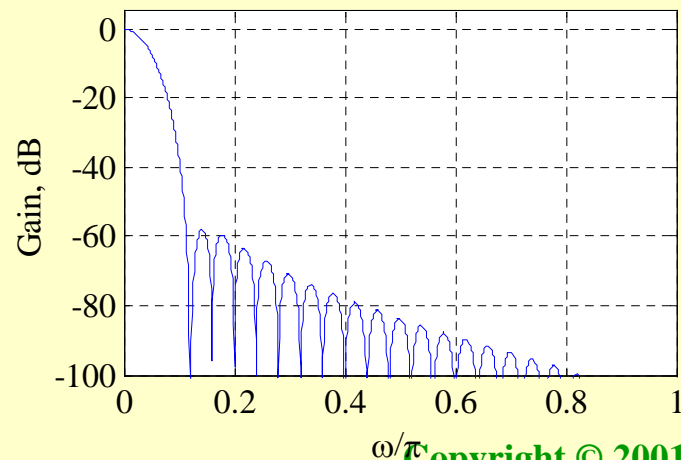
Rectangular window

Hanning window

Hamming window

Blackman window

Fixed Window Functions

- Magnitude spectrum of each window characterized by a main lobe centered at $\omega = 0$ followed by a series of sidelobes with decreasing amplitudes

- Parameters predicting the performance of a window in filter design are:

- **Main lobe width**

- **Relative sidelobe level**

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Fixed Window Functions

- **Main lobe width** Δ_{ML} - given by the distance between zero crossings on both sides of main lobe

- **Relative sidelobe level** $A_{s\ell}$ - given by the difference in dB between amplitudes of largest sidelobe and main lobe

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Fixed Window Functions
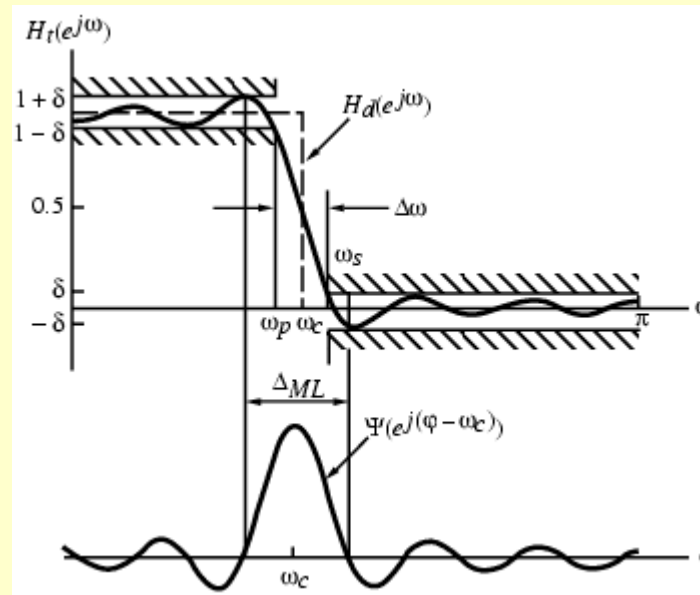
- Observe $H_t(e^{j(\omega_c+\Delta\omega)}) + H_t(e^{j(\omega_c-\Delta\omega)}) \cong 1$

- Thus, $H_t(e^{j\omega_c}) \cong 0.5$

- Passband and stopband ripples are the same

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Fixed Window Functions

- Distance between the locations of the maximum passband deviation and minimum stopband value $\cong \Delta_{ML}$

- Width of transition band

$$\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$$

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Fixed Window Functions

- To ensure a fast transition from passband to stopband, window should have a very small main lobe width

- To reduce the passband and stopband ripple δ, the area under the sidelobes should be very small

- Unfortunately, these two requirements are contradictory

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Fixed Window Functions

- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency ω_c, and is essentially constant

- In addition,

$$\Delta\omega \approx \frac{c}{M}$$

where c is a constant for most practical purposes

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Fixed Window Functions

- **Rectangular window** - $\Delta_{ML} = 4\pi/(2M+1)$

  $A_{s\ell} = 13.3\,\text{dB}, \; \alpha_s = 20.9\,\text{dB}, \; \Delta\omega = 0.92\pi/M$

- **Hann window** - $\Delta_{ML} = 8\pi/(2M+1)$

  $A_{s\ell} = 31.5\,\text{dB}, \; \alpha_s = 43.9\,\text{dB}, \; \Delta\omega = 3.11\pi/M$

- **Hamming window** - $\Delta_{ML} = 8\pi/(2M+1)$

  $A_{s\ell} = 42.7\,\text{dB}, \; \alpha_s = 54.5\,\text{dB}, \; \Delta\omega = 3.32\pi/M$

- **Blackman window** - $\Delta_{ML} = 12\pi/(2M+1)$

  $A_{s\ell} = 58.1\,\text{dB}, \; \alpha_s = 75.3\,\text{dB}, \; \Delta\omega = 5.56\pi/M$

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Fixed Window Functions

- Filter Design Steps -

  (1) Set

$$\omega_c = (\omega_p + \omega_s)/2$$

  (2) Choose window based on specified α_s
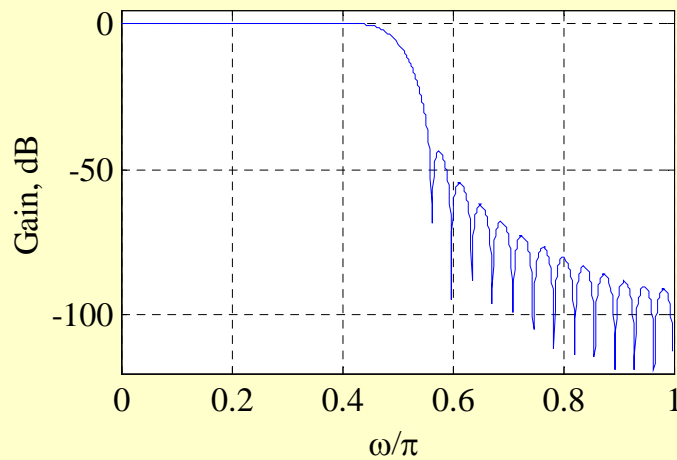
  (3) Estimate M using

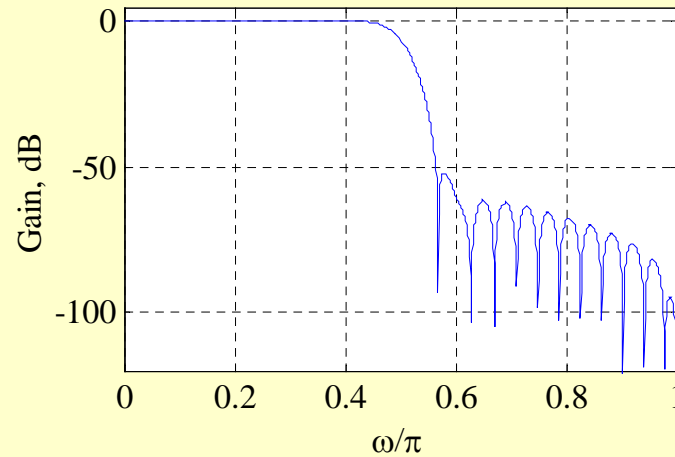$$\Delta\omega \approx \frac{c}{M}$$

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FIR Filter Design Example

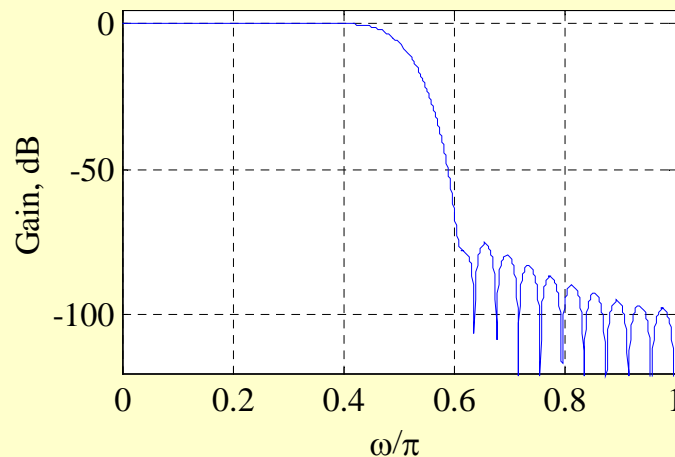- Lowpass filter of length 51 and $\omega_c = \pi/2$

Lowpass Filter Designed Using Hann window

Lowpass Filter Designed Using Hamming window

Lowpass Filter Designed Using Blackman window

FIR Filter Design Example

- An increase in the main lobe width is associated with an increase in the width of the transition band

- A decrease in the sidelobe amplitude results in an increase in the stopband attenuation

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Adjustable Window Functions

- **Dolph-Chebyshev Window** -

$$w[n] = \frac{1}{2M+1}\left[\frac{1}{\gamma} + 2\sum_{k=1}^{M} T_k(\beta \cos\frac{k}{2M+1})\cos\frac{2nk\pi}{2M+1}\right],$$
$$-M \leq n \leq M$$

where
$$\gamma = \frac{\text{amplitude of sidelobe}}{\text{main lobe amplitude}}$$

$$\beta = \cosh(\frac{1}{2M}\cosh^{-1}\frac{1}{\gamma})$$

and
$$T_\ell(x) = \begin{cases} \cos(\ell \cos^{-1} x), & |x| \leq 1 \\ \cosh(\ell \cosh^{-1} x), & |x| > 1 \end{cases}$$

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Adjustable Window Functions

- Dolph-Chebyshev window can be designed with any specified relative sidelobe level while the main lobe width adjusted by choosing length appropriately
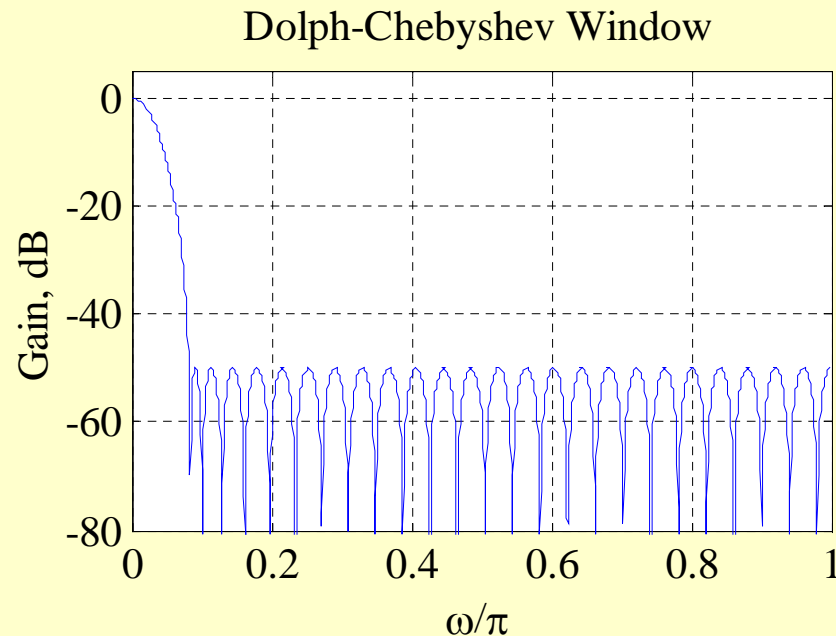
- Filter order is estimated using

$$N = \frac{2.056\alpha_s - 16.4}{2.85(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth, e.g, for a lowpass filter

$$\Delta\omega = \omega_s - \omega_p$$

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Adjustable Window Functions

- Gain response of a Dolph-Chebyshev window of length 51 and relative sidelobe level of 50 dB is shown below

Dolph-Chebyshev Window

Adjustable Window Functions

Properties of Dolph-Chebyshev window:

- All sidelobes are of equal height

- Stopband approximation error of filters designed have essentially equiripple behavior

- For a given window length, it has the smallest main lobe width compared to other windows resulting in filters with the smallest transition band

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Adjustable Window Functions

- **Kaiser Window** -

$$w[n] = \frac{I_0\{\beta\sqrt{1-(n/M)^2}\}}{I_0(\beta)}, \quad -M \leq n \leq M$$

where β is an adjustable parameter and $I_0(u)$ is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty} [\frac{(u/2)^r}{r!}]^2$$

- Note $I_0(u) > 0$ for $u > 0$

- In practice $I_0(u) \cong 1 + \sum_{r=1}^{20} [\frac{(u/2)^r}{r!}]^2$

Adjustable Window Functions

- β controls the minimum stopband attenuation of the windowed filter response

- β is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth

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FIR Filter Design Example

- Specifications: $\omega_p = 0.3\pi$, $\omega_s = 0.5\pi$, $\alpha_s = 40$ dB

- Thus $\omega_c = (\omega_p + \omega_s)/2 = 0.4\pi$

$$\delta_s = 10^{-\alpha_s/20} = 0.01$$

$$\beta = 0.5842(19)^{0.4} + 0.07886 \times 19 = 3.3953$$

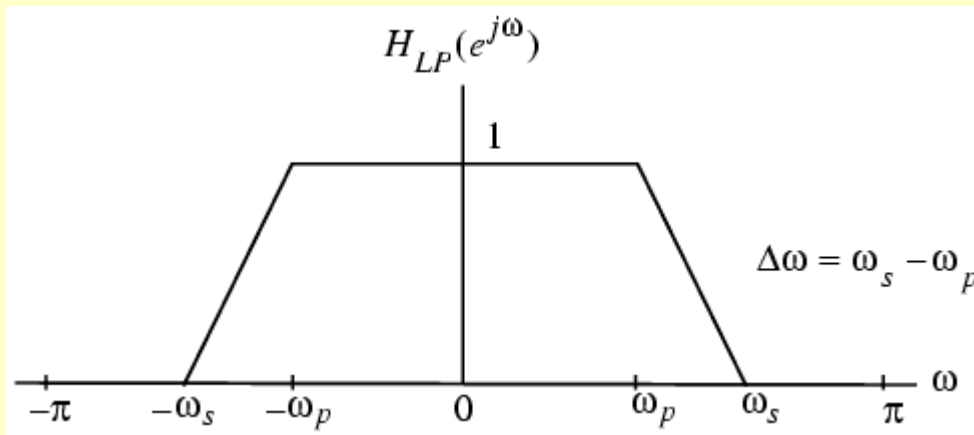$$N = \frac{32}{2.285(0.2\pi)} = 22.2886$$

- Choose $N = 24$ implying $M = 12$

FIR Filter Design Example

- Hence $h_t[n] = \dfrac{\sin(0.4\pi n)}{\pi n} \cdot w[n], \quad -12 \leq n \leq 12$

  where $w[n]$ is the n-th coefficient of a length-25 Kaiser window with $\beta = 3.3953$

Kaiser Window

Lowpass filter designed with Kaiser window
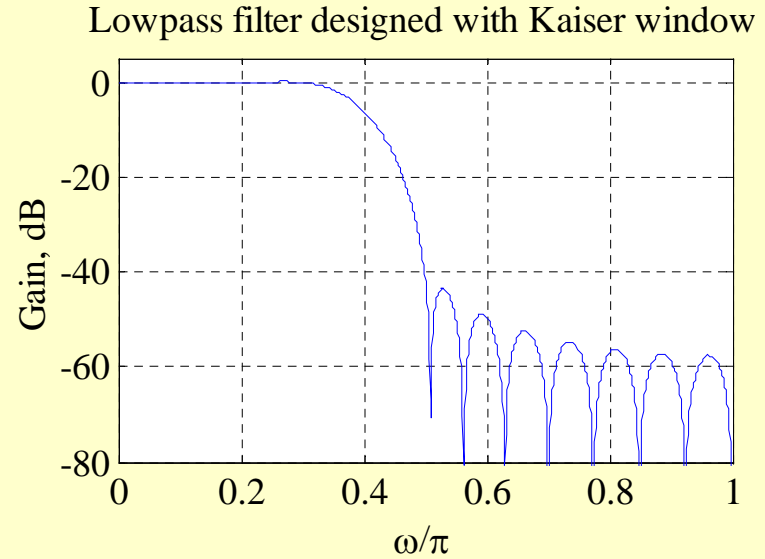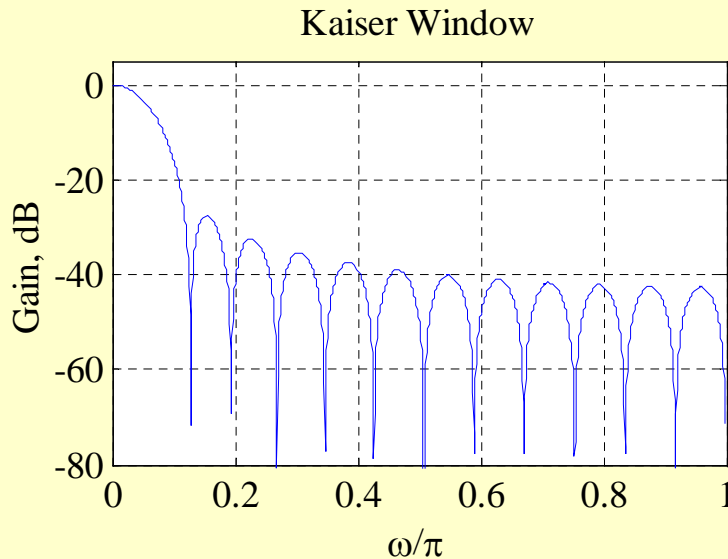
Impulse Responses of FIR Filters with a Smooth Transition

- First-order spline passband-to-stopband transition

$$\omega_c = (\omega_p + \omega_s)/2$$

$$\Delta\omega = \omega_s - \omega_p$$

$$h_{LP}[n] = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \cdot \dfrac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

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Impulse Responses of FIR Filters with a Smooth Transition

- Pth-order spline passband-to-stopband transition

$$h_{LP}[n] = \begin{cases} \omega_c / \pi, & n = 0 \\ \left(\dfrac{2\sin(\Delta\omega n / 2P)}{\Delta\omega n / 2P} \right)^P \cdot \dfrac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

Lowpass FIR Filter Design Example

- Example