Module 9 : Numerical Relaying II : DSP Perspective

Lecture 35 : Computation of Phasor from Discrete Fourier Transform

Objectives

In this lecture, we will

- Develop methodology to evaluate phasor from DFT.
- The method would be generalized to handle moving windows.
- Recursive forms of DFT approach will be derived.

35.1 DFT of a Sinusoid

Consider a sinusoidal input signal of frequency ω_0, given by

$$x(t) = \sqrt{2}X\sin(\omega_0 t + \phi) \tag{1}$$

This signal is conveniently represented by a phasor $\overline{X}$

$$\overline{X} = Xe^{j\phi} = X(\cos\phi + j\sin\phi) \tag{2}$$

Assume that $x(t)$ is sampled N times per cycle such that $T_0 = Nt_s$. Then,

$$x_k = \sqrt{2}X\sin(\frac{2\pi}{N}k + \phi) \tag{3}$$

and $f_s = Nf_0$. In the transform domain, transformed components are separated by f_s/N. Thus, choice m = 1 corresponds to extracting the fundamental frequency component. The Discrete Fourier Transform of x_k contains the fundamental frequency component given by

$$X_1 = \frac{2}{N}\sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}k} \tag{4}$$

$$= \frac{2}{N}\sum_{k=0}^{N-1} x_k \cos(\frac{2\pi}{N}k) - j\frac{2}{N}\sum_{k=0}^{N-1} x_k \sin(\frac{2\pi}{N}k)$$

$$= X_c - jX_s \tag{5}$$

where, $X_c = \frac{2}{N}\sum_{k=0}^{N-1} x_k \cos(\frac{2\pi}{N}k)$ (6)

35.1 DFT of a Sinusoid (contd..)

$$X_s = \frac{2}{N}\sum_{k=0}^{N-1} x_k \sin(\frac{2\pi}{N}k) \tag{7}$$

Substituting x_k from (3) in (6) and (7) it can be shown that for a sinusoidal input signal given by (1)

$$X_c = \sqrt{2}X\sin\phi \qquad X_s = \sqrt{2}X\cos\phi \tag{8}$$

Therefore, from equation (2), (5) and (8) it follows that

$$\overline{X} = \frac{1}{\sqrt{2}} jX_1 = \frac{1}{\sqrt{2}}\left(X_s + jX_c\right) \tag{9}$$

When the input signal contains other frequency components as well, the phasor calculated by equation (9) is a filtered fundamental frequency phasor. It is presumed here that the input signal must be band-limited to satisfy the "Nyquist Criterion", to avoid errors due to aliasing affects.

35.2 Phase Computation using DFT

In relaying applications, during steady state, we will be working with a moving window, such that each window is having N-most recent samples. Let X_c^w and X_s^w indicate X_c and X_s component of DFT for w^{th} window. By our convention, window number is the sample number of the first sample in the active window. As the first N - 1 windows are incomplete, there window number is one. Afterwards, window number is incremented by one for each new sample.

$$X_c^w = \frac{2}{N}\sum_{k=0}^{N-1} x_{k+w} \cos\frac{2\pi}{N}k$$

$$= \frac{2}{N}\sum_{k=0}^{N-1} \sqrt{2}X \sin\left(\frac{2\pi}{N}\overline{k+w} + \phi\right)\cos\frac{2\pi}{N}k$$

$$= \frac{\sqrt{2}X}{N}\sum_{k=0}^{N-1}\left\{\sin\left(\frac{4\pi}{N}k + \frac{2\pi}{N}w + \phi\right) + \sin\left(\frac{2\pi}{N}w + \phi\right)\right\}$$

$$= \frac{\sqrt{2}X}{N}\left\{0 + N\sin\left(\frac{2\pi}{N}w + \phi\right)\right\} = \sqrt{2}X\sin\left(\frac{2\pi}{N}w + \phi\right)$$

35.2 Phase Computation using DFT (contd..)

Similarly,

$$X_s^w = \frac{2}{N}\sum_{k=0}^{N-1} x_{k+w} \sin\frac{2\pi}{N}k$$

$$= \frac{2}{N}\sum_{k=0}^{N-1} \sqrt{2}X \sin\left(\frac{2\pi}{N}\overline{k+w} + \phi\right)\sin\frac{2\pi}{N}k$$

$$= \frac{\sqrt{2}X}{N}\sum_{k=0}^{N-1}\left\{\cos\left(\frac{2\pi}{N}w + \phi\right) - \cos\left(\frac{4\pi}{N}k + \frac{2\pi}{N}w + \phi\right)\right\}$$

$$= \frac{\sqrt{2}XN}{N}\cos\left(\frac{2\pi}{N}w + \phi\right) = \sqrt{2}X\cos\left(\frac{2\pi}{N}w + \phi\right)$$

Hence, DFT estimated at w^{th} window for (m = 1) is given by

$$X^w(1) = X_c^w - jX_s^w = \sqrt{2}X\left(\sin\left(\frac{2\pi}{N}w + \phi\right) - j\cos\left(\frac{2\pi}{N}w + \phi\right)\right)$$

Hence, phasor estimate for w^{th} window is given by

$$\overline{X}^w = \frac{1}{\sqrt{2}}\left(X_s^w + jX_c^w\right) = X\left(\cos\left(\frac{2\pi}{N}w + \phi\right) + j\sin\left(\frac{2\pi}{N}w + \phi\right)\right)$$

$$= X e^{j\left(\frac{2\pi}{N}w+\phi\right)}$$

$$= X e^{j\phi} e^{j\frac{2\pi}{N}w}$$

Thus, we see that with moving window, the phasor estimate rotates at a speed of $e^{j\frac{2\pi}{N}}$ per window. This rotation in phasor during computation can be directly derived from DFT – phase shifting property.

35.2Phase Computation using DFT (contd..)

In the computation of phasor, we would not like this phasor rotation to occur due to DFT phase shifting. This can be avoided if we modify equation (6) and (7). Replace k = 0 by k = w , in the lower limit and k = N + w - 1 in the upper limit, we get

$$X(m) = \frac{2}{N}\sum_{k=w}^{k+w-1} x_k \, e^{\frac{-j2\pi km}{N}} = X_c - jX_s$$

Where $X_c^w(m) = \frac{2}{N}\sum_{k=w}^{N+w-1} x_k \cos\left(\frac{2\pi km}{N}\right)$ (10)

$$X_s^w(m) = \frac{2}{N}\sum_{k=w}^{N+w-1} x_k \sin\left(\frac{2\pi km}{N}\right) \quad (11)$$

Substituting x_k in equation (10), we get

$$X_c^w = \frac{2\sqrt{2}\,|X|}{N}\frac{1}{2}\sum_{k=w}^{N+w-1} 2\sin\left(\frac{2\pi k}{N}+\phi\right)\cos\left(\frac{2\pi nm}{N}\right)$$

$$= \frac{2\sqrt{2}\,|X|}{N}\frac{1}{2}\sum_{k=w}^{N+w-1}\left[\sin\left(\frac{2\pi k}{N}(1-m)+\phi\right)+\sin\left(\frac{2\pi k}{N}(1+m)+\phi\right)\right] = \sqrt{2}\,|X|\sin\phi \text{ for m = 1;}$$

$$X_s^w = \frac{2\sqrt{2}\,|X|}{N}\frac{1}{2}\sum_{k=w}^{N+w-1} 2\sin\left(\frac{2\pi k}{N}+\phi\right)\sin\left(\frac{2\pi km}{N}\right)$$

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Hence, $\overline{X}^w = \frac{1}{\sqrt{2}}\left(X_s^w + jX_s^w\right) = X$

Thus, we conclude that artificial rotation induced in phasor computation by using DFT equations can be eliminated by appropriately modifying the offset in the summation index of DFT equation.

35.3Full Cycle Recursive DFT

From equation (10) and (11), we have

$$X_c^w(m) = \frac{2}{N}\sum_{k=w}^{N+w-1} x_k \cos\left(\frac{2\pi km}{N}\right)$$

and

$$X_s^w(m) = \frac{2}{N}\sum_{k=w}^{k=N+w-1} x_k \sin\left(\frac{2\pi km}{N}\right)$$

35.2 Phase Computation using DFT (contd..)

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$$X_c^w = \frac{2\sqrt{2}X}{N}\frac{1}{2}\sum_{k=w}^{N+w-1} 2\sin\left(\frac{2\pi k}{N}+\phi\right)\cos\left(\frac{2\pi nm}{N}\right)$$

$$= \frac{2\sqrt{2}X}{N}\frac{1}{2}\sum_{k=w}^{N+w-1}\left[\sin\left(\frac{2\pi k}{N}(1-m)+\phi\right)+\sin\left(\frac{2\pi k}{N}\right)(1+m)+\phi\right] = \sqrt{2}X\sin\phi \text{ for } m = 1;$$

$$X_s^w = \frac{2\sqrt{2}X}{N}\frac{1}{2}\sum_{k=w}^{N+w-1} 2\sin\left(\frac{2\pi k}{N}+\phi\right)\sin\left(\frac{2\pi km}{N}\right)$$

$$= \frac{2\sqrt{2}X}{N}\frac{1}{2}\sum_{k=w}^{N+w-1}\left[\cos\left(\frac{2\pi k}{N}(1-m)+\phi\right)-\cos\left(\frac{2\pi k}{N}(1+m)+\phi\right)\right] = \sqrt{2}X\cos\phi \text{ for } m = 1;$$

Hence, $\overline{X}^w = \frac{1}{\sqrt{2}}\left(X_s^w + jX_s^w\right) = X$

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and

$$X_s^w(m) = \frac{2}{N}\sum_{k=w}^{k=N+w-1} x_k \sin\frac{2\pi km}{N}$$

35.3 Full Cycle Recursive DFT (contd..)

The computation for X_c^w can be visualized from the following pair wise multiplication and add sequence enclosed in red brackets in the fig 35.1.

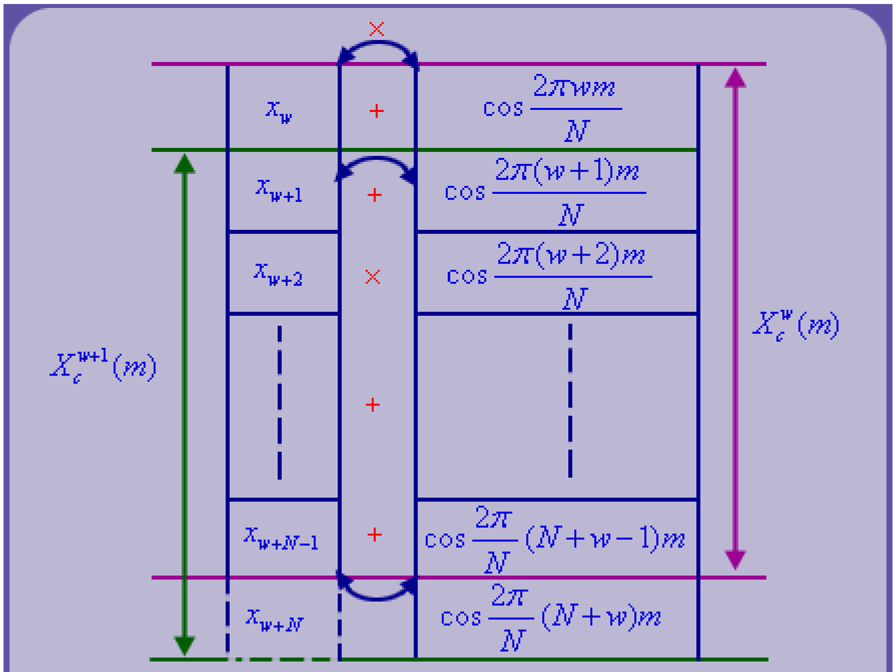

Fig 35.1 Mechanism of Recursive DFT

Now, to compute $X_c^{w+1}(m)$, we add one new sample x_{w+N}. Computation of $X_c^{w+1}(m)$ from $X_c^{w}(m)$ is shown by pairing with in the green boundary. It is now obvious, that

$$X_c^{w+1}(m) = X_c^{w}(m) + x_{w+N} \cos\left(\frac{2\pi}{N}(N+w)m\right) - x_w \cos\left(\frac{2\pi}{N} wm\right).$$

Since,

$$\cos\left(\frac{2\pi}{N}(N+w)m\right) = \cos\left(2\pi m + \frac{2\pi}{N} wm\right) = \cos\left(\frac{2\pi}{N} wm\right)$$

for m - an integer. Thus,

$$X_c^{w+1}(m) = X_c^{w}(m) + \left(x_{w+N} - x_w\right) \cos\left(\frac{2\pi}{N} wm\right) \tag{12}$$

Similarly, for sine component, we have

$$X_s^{w+1}(m) = X_s^{w}(m) + [x_{w+N} - x_w] \sin\left(\frac{2\pi}{N} wm\right) \tag{13}$$

35.3 Full Cycle Recursive DFT (contd..)

Eqns. 12 and 13 provide a recursive update for DFT computation. The advantage of recursive form is that it reduces computation from 2N multiply add operation in normal DFT to 4 additions and 2 multiplications.

To begin with, we will get 2N set $X_s^{w}(m) = X_s^{w}(m) = 0$. When the first window is full populated with N-samples, we will get correct values of $X_c(m)$ and $X_m(m)$. Afterwards, for a stationary phasor at fundamental frequency $x_{w+N} = x_w$ and hence, the DFT latches on to the appropriate phasor.

35.4 Half Cycle DFT Form for Phase Estimation

If our primary interest is to extract fundamental phasor component in the signal then, it can be verified that, restricting moving window to half a cycle does not alter the end result of eqn. (5) and (6) provided that N-now represents, number of samples per half-cycle. Now, the sample x_k is given by

$x_k = \sqrt{2} \times \sin\left(\frac{\pi}{N} k + \phi\right)$. Thus, half cycle form of DFT phasor estimation is given by the following eqns.

$$X_c^w(m) = \frac{2}{N}\sum_{k-w}^{N+w-1} x_k \cos\left(\frac{\pi}{N}k\right) \qquad (14)$$

$$X_s^w = \frac{2}{N}\sum_{k-w}^{N+w-1} x_k \sin\left(\frac{\pi}{N}k\right) \qquad (15)$$

Advantage of half cycle algorithm is that the moving window latches on to the post fault signal in $\frac{1}{2}$ of a cycle. Thus, compared to full cycle version, it is twice as fast. A keen observer would have noticed that DFT based on moving window phasor estimation equations are identical to the full cycle and cycle fourier algorithms derived in lecture-31. Thus, the frequency response of fourier algorithms developed in lecture 31 applies to the DFT version. In particular, it is not surprising to see that harmonic rejection property of half cycle algorithm is inferior to its full cycle avatar. This is consistent with the 'speed vs accuracy' conflict, we have discussed earlier.

35.5 Half Cycle Recursive DFT

Recursive form of half cycle DFT can be derived in an analogous manner to full cycle DFT. Realizing that

$$\cos\left(\frac{\pi}{N}(N+w)m\right) = \cos\left(\pi m + \frac{\pi}{N}mw\right)$$

$$= -\cos\left(\frac{\pi}{N}mw\right) \leftarrow \text{m odd}$$

$$\sin\left(\frac{\pi}{N}(N+w)m\right) = \sin\left(\pi m + \frac{\pi}{N}mw\right)$$

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we get the following recursive update forms for fundamental phasor computation.

$$X_c^{w+1} = X_c^w - (x_{w+N} + x_w)\cos\left(\frac{\pi}{N}w\right)$$

$$X_s^{w+1} = X_s^w - (x_{w+N} + x_w)\sin\left(\frac{\pi}{N}w\right)$$

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$$X_c^{w+1} = X_c^w - (x_{w+N} + x_w)\cos\frac{\pi}{N}w$$

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35.3 Full Cycle Recursive DFT (contd..)

Eqns. 12 and 13 provide a recursive update for DFT computation. The advantage of recursive form is that it reduces computation from 2N multiply add operation in normal DFT to 4 additions and 2 multiplications.

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we get the following recursive update forms for fundamental phasor computation.

$$X_c^{w+1} = X_c^w - (x_{w+N} + x_w)\cos\frac{\pi}{N}w$$

$$X_s^{w+1} = X_s^w - (x_{w+N} + x_w)\sin\frac{\pi}{N}w$$

Review Questions

1. Consider, the following signal

$$x^{(t)} = 10\sin(2\pi\times 50\times t + 30^\circ)$$

(a) Generate samples on this waveform using sampling frequency of 12 samples per cycle.

(b) Apply full cycle and half cycle algorithms to estimate phasor from generalized DFT approach.

(c) Repeat (b) using recursive forms.

2. Now consider the following signal.

$$x(t) = 10\sin(2\pi\times 50\times t + 30^\circ) + 5\sin(4\pi\times 50\times t + 17.5^\circ) + \frac{10}{3}\cos(6\pi\times 50\times t + 75^\circ)$$

Repeat (2) and comment on the accuracy of full cycle and half cycle estimation methods.

Recap

In this lecture we have learnt the following:

- Computation of phasor from DFT.
- Computation of stationary phasors from moving windows.
- Recursive DFT approach (full cycle and half cycle).