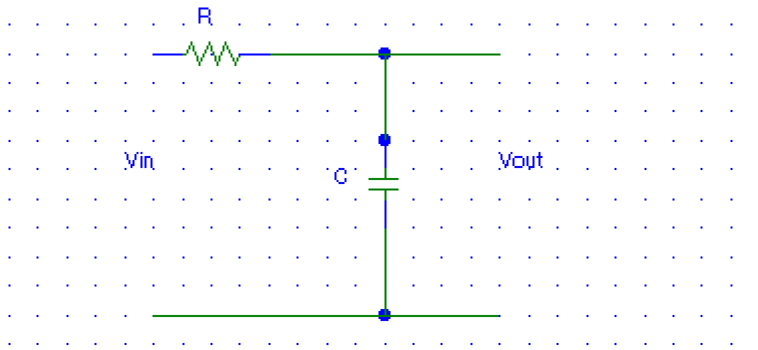
Solve the Simple RC low-pass using Differential Equations:

Here is the circuit diagram



The Kirchoff loop equation is:

$$v_{in}(t) = R * i(t) + \frac{1}{C}\int i(t)dt$$

First solve the Homogeneous Equation to get the Homogeneous "Natural" solution:

$$R * i(t) + \frac{1}{C}\int i(t)dt = 0$$

Differentiating both sides of the equation and multiplying by C

$$RC * \frac{di}{dt} + i = 0$$

The solution to this equation is of the form $K_H \varepsilon^{-at}$ and substituting:

$$RC * \left(-K_H a\varepsilon^{-at}\right) + \left(K_H \varepsilon^{-at}\right) = 0$$

$$\left(K_H \varepsilon^{-at}\right) = RC * \left(K_H a\varepsilon^{-at}\right)$$

Simplifying,

$\varepsilon^{-at} = RC * a\varepsilon^{-at}$ or a = 1/RC

The **Homogeneous Solution** is therefore:

$$i_H(t) = K_H \varepsilon^{-\frac{t}{RC}}$$

But $i_H(0) = \frac{V_{in}}{R}$ since the voltage across the capacitor starts at 0 and,

$$i_H(t) = \frac{V_{in}}{R}\varepsilon^{-\frac{t}{RC}}$$

But we want the voltage out which is:

$$v_H(t) = \frac{1}{C}\int \frac{V_{in}}{R}\varepsilon^{-\frac{t}{RC}}dt$$

Substituting,

$v_H(t) = \frac{V_{in}}{RC} * (-RC)\varepsilon^{-\frac{t}{RC}} + k = -V_{in} * \varepsilon^{-\frac{t}{RC}} + k$ where k is the constant of integration

Now we need to find the **Particular Solution** that is due to the **Forcing Function** (input)

Case 1 ($t < 0$): obviously, the output is again zero.

Case 2 ($0 < t < 1$)

We have that the original input is a constant "V_{in}" which was differentiated and became 0

The output needs to be of the form

$$i_p(t) = A + B * t$$

Substituting into our original differential equation:

$$RC * \frac{di}{dt} + i = 0$$

Or,

$$RC * (B) + (A + B * t) = 0$$

Since this must be true for all $0 < t < 1$, $B = 0$ (from the t term) and from the constant term A also is 0 so

$$i_p(t) = 0$$

Therefore the total solution is

$v(t) = -V_{in} * \varepsilon^{-\frac{t}{RC}} + k$ but this must be zero at $t = 0$

$$0 = -V_{in} * \varepsilon^{-\frac{0}{RC}} + k = -V_{in} * 1 + k$$

$k = V_{in}$

$v(t) = V_{in} * \left(1 - \varepsilon^{-\frac{t}{RC}}\right)$ for $0 < t < 1$