EE 102 # Homework #5 Solutions **Fall 2001**

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1. The transfer function of the RC circuit can be found by Laplace transforming the impulse response function from Lecture 4,

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \implies H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{1}{sRC + 1}.$$

Alternatively, if you've taken circuit courses with Laplace you could use the voltage division formula to get directly

$$Y(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \cdot X(s) = \frac{1}{sRC + 1} X(s).$$

Plugging in $X(s) = \frac{\omega_0}{s^2 + \omega_0^2}$ and using the fact that $\omega_0 = \frac{1}{RC}$ we have

$$Y(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \cdot \frac{\omega_0}{s^2 + \omega_0^2} = \frac{\omega_0}{s + \omega_0} \cdot \frac{\omega_0}{s^2 + \omega_0^2} = \frac{a_0}{s + \omega_0} + \frac{a_1 s + a_2}{s^2 + \omega_0^2}.$$

Solving for a_0, a_1, and a_2 we get the equation

$$\omega_0^2 = s^2(a_0 + a_1) + s(a_1 \omega_0 + a_2) + a_0 \omega_0^2 + a_2 \omega_0,$$

and therefore $a_0 = -a_1$, $a_1 = -a_2/\omega_0$, and $a_2 = \frac{1}{2}\omega_0$, which yields $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{2}$, and $a_2 = \frac{1}{2}\omega_0$. With these values we get

$$Y(s) = \frac{1}{2}\left(\frac{1}{s + \omega_0} - \frac{s}{s^2 + \omega_0^2} + \frac{\omega_0}{s^2 + \omega_0^2}\right).$$

Transforming back to time-domain finally gives

$$y(t) = \frac{1}{2} u(t) \left[e^{-\omega_0 t} - \cos(\omega_0 t) + \sin(\omega_0 t) \right].$$

2. Given the pole information we can write the denominator as

$$(s - (-1 + i))(s - (-1 - i)) = (s + 1)^2 + 1 = s^2 + 2s + 2$$

A *proper* transfer function has the same degree for s in the numerator as in the denominator, so the numerator is of the form

$$a_2 s^2 + a_1 s + a_0,$$

which gives

$$H(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + 2s + 2}.$$

Note that the system is LTI, stable (left half-plane poles). So we know that:

- For an input $x(t) = e^{i\omega_0 t}$, the output is $y(t) = H(i\omega_0)e^{i\omega_0 t}$.
- For $x(t) = \cos(\omega_0 t)$, the output is $y(t) = |H(i\omega_0)|\cos(\omega_0 t + \phi)$.

(a) For $x(t) \equiv 1 = e^{i0t}$, $\omega_0 = 0$, and $y(t) \equiv 2$. Therefore $H(i0) = 2$.
For $x(t) = \cos(2t)$, $\omega_0 = 2$, and $y(t) = 0$. Therefore $|H(i2)| = 0$.
From this information we get

$$H(i0) = \frac{a_0}{2} = 2,$$

which yields $\underline{a_0 = 4}$. We also have

$$|H(i2)| = |\frac{-4a_2 + a_0 + i2a_1}{-4 + i4 + 2}| = 0,$$

and by setting real and imaginary parts of the numerator to zero separately we get $-4a_2 + 4 = 0$ and $i2a_1 = 0$. Therefore $\underline{a_2 = 1}$ and $\underline{a_1 = 0}$. The final result is

$$H(s) = \frac{s^2 + 4}{s^2 + 2s + 2}.$$

(b) We transform $x(t)$ to $X(s)$ and use the following equation

$$Y(s) = H(s)X(s) = \frac{s^2 + 4}{s^2 + 2s + 2}\frac{2}{s^2 + 4} = \frac{2}{s^2 + 2s + 2} = 2\frac{1}{(s+1)^2 + 1}.$$

Transforming back to time-domain gives

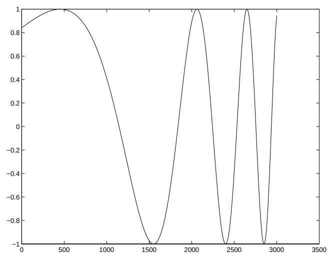
$$y(t) = 2e^{-t}\sin(t)u(t).$$

3. (a) $\cos(4t) + \sin(3t - 3)$ is periodic with period 2π. Explanation: $\cos(4t)$ is periodic with period $\frac{\pi}{2}$. $\sin(3t - 3)$ is periodic with period $\frac{2\pi}{3}$, the phase shift does not influence the period. The sum of two periodic functions is periodic with the period being the least common multiple of the periods of each term in the sum.

(b) $e^{\sin(t)}$ is periodic with period 2π. Explanation: $\sin(t)$ is periodic with period 2π. The exponential of a periodic function is also periodic with the same period:

$$e^{\sin(t+2\pi)} = e^{\sin(t)}.$$

(c) $\sin(e^t)$ is not periodic. See graph.



4. With $T = 4$ and $\omega_0 = \frac{\pi}{2}$ we get

$$\begin{aligned} F_n &= \tfrac{1}{4}\int_0^2 2\cos(\tfrac{\pi}{2}t)e^{-in\frac{\pi}{2}t}dt \\ &= \tfrac{1}{4}\int_0^2 (e^{i\frac{\pi}{2}t} + e^{-i\frac{\pi}{2}t})e^{-in\frac{\pi}{2}t}dt \\ &= \tfrac{1}{4}\int_0^2 \left[e^{i\frac{\pi}{2}t(1-n)} + e^{-i\frac{\pi}{2}t(1+n)}\right] dt \\ &= \tfrac{1}{4}\left[\frac{e^{i\frac{\pi}{2}t(1-n)}}{i\frac{\pi}{2}(1-n)}\right]_0^2 - \tfrac{1}{4}\left[\frac{e^{-i\frac{\pi}{2}t(1+n)}}{i\frac{\pi}{2}(1+n)}\right]_0^2 \\ &= \frac{1}{2\pi i}\left[\frac{e^{i\pi(1-n)}-1}{1-n} - \frac{e^{-i\pi(1+n)}-1}{1+n}\right] \end{aligned}$$

The above derivation assumes that $n \neq \pm 1$. Knowing that $e^{i\pi} = e^{-i\pi} = -1$ and $(-1)^n = (-1)^{-n}$ we get

$$\begin{aligned} F_n &= \frac{1}{2\pi i}\left[\frac{(-1)(-1)^n-1}{1-n} - \frac{(-1)(-1)^n-1}{n+1}\right] \\ &= \frac{1}{2\pi i}\left[\frac{(-1)^n+1}{n-1} + \frac{(-1)^n+1}{n+1}\right] \\ &= \frac{(-1)^n+1}{2\pi i}\left[\frac{1}{n-1} + \frac{1}{n+1}\right] \\ &= -\frac{in((-1)^n+1)}{\pi(n^2-1)} \qquad = \begin{cases} 0 & \text{if } n \text{ odd}, n \neq \pm 1 \\ -\frac{2ni}{\pi(n^2-1)} & \text{if } n \text{ even} \end{cases} \end{aligned}$$

For $n = 1$ we get

$$\begin{aligned} F_1 &= \tfrac{1}{4}\int_0^2 (e^{i\frac{\pi}{2}t} + e^{-i\frac{\pi}{2}t})e^{-i\frac{\pi}{2}t}dt \\ &= \tfrac{1}{4}\int_0^2 (1 + e^{-i\pi t})dt \\ &= \tfrac{1}{4}\left[[t]_0^2 + [\tfrac{e^{-i\pi t}}{-i\pi}]_0^2\right] \\ &= \tfrac{1}{4}\left[2 + \tfrac{1}{-i\pi} - \tfrac{1}{-i\pi}\right] \\ &= \tfrac{1}{2} \end{aligned}$$

Similarly we obtain

$$F_{-1} = \frac{1}{2}$$