MODELLING IN THE TIME DOMAIN

In this lesson you will learn the following :

- How to find a mathematical model, called a *state-space* representation, for a linear time invariant system
- How to convert between transfer function and *state-space* models
- How to linearize a *state-space* representation

Two approaches are available for the analysis and design of feedback control systems. The first is known as the ***classical*** or ***frequency-domain*** technique. This approach is based on converting a system's differential equation to a transfer function. The primary disadvantage of the classical approach is its limited applicability. It can be applied only linear time-invariant systems. But this approach rapidly provides stability and transient response information.

The ***state-space*** approach (also referred to as the ***modern*** or ***time-domain*** approach) is a unified method for modeling, analyzing and designing a wide range of systems. We can use the state-space approach both linear and nonlinear systems. Also it can handle the systems with nonzero initial conditions.

In the state-space analysis, we select a particular ***subset*** of all possible system variables and call the variables in this subset ***state variables***.

For an *n*th-order system, we write *n* ***simultaneous, first order differential equations*** in terms of the state variables. We call this system simultaneous differential equations ***state equations***.

We algebrically combine the state variables with the system's input and find all of the other system variables for $t \geq t_0$, we call this algebraic equation the ***output equation***.

We consider the state equations and the output equations a viable representation of the system. We call this representation of the system a ***state-space represantation***.

Let us now follow the steps for state-space represantationthrough an example. Consider RL network shown in figure with an initial current of i(0).

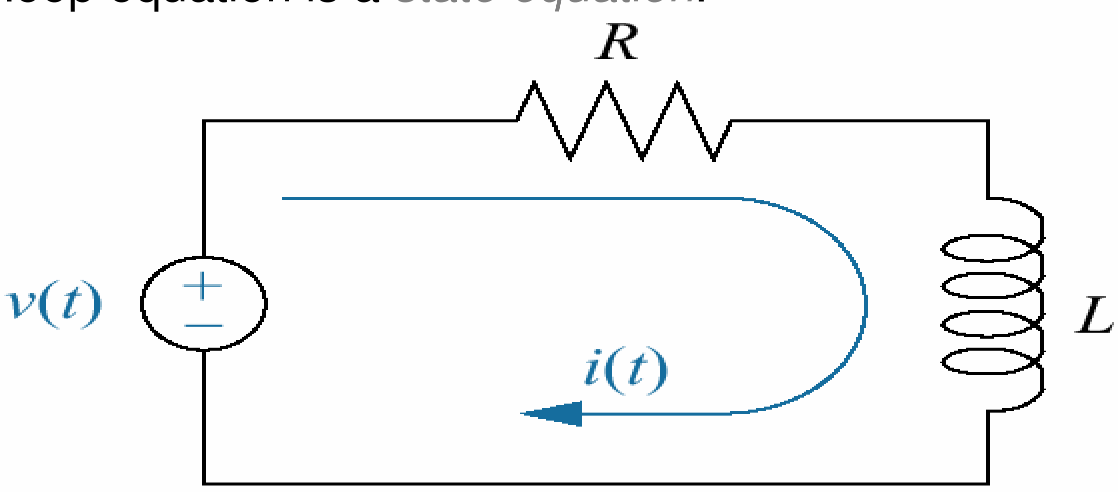
1.) Select the current as state variable

2.)Write the loop equaiton $L\dfrac{di}{dt}+Ri=v(t)$ and 3.) Take the Laplace transform with including the initial conditions : L[sI(s)-i(0)]+RI(s) = V(s)

Assuming the input, v(t), to be a unit step, u(t), whose Laplace transform is V(s)=1/s, we solve for I(s) and get

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+\frac{R}{L}}\right)+\frac{i(0)}{s+\frac{R}{L}} \Rightarrow i(t)=\frac{1}{R}\left(1-e^{-\left(\frac{R}{L}\right)t}\right)+i(0)e^{-\left(\frac{R}{L}\right)t}$$

The funtion i(t) is a subset of all possible network variables that we are able to find from yhe last equation if we know its initial condition, i(0), and the input v(t). Thus i(t) is a *state variable*, and the loop equation is a *state equation*.



4.) We can solve for all of the other network variables *algebraically* in terms of i(t) and v(t). For example the voltage across the resistor is $V_R(t) = Ri(t)$. The voltage across the inductor is $V_L(t) = V – Ri(t)$. The derivative of the current is

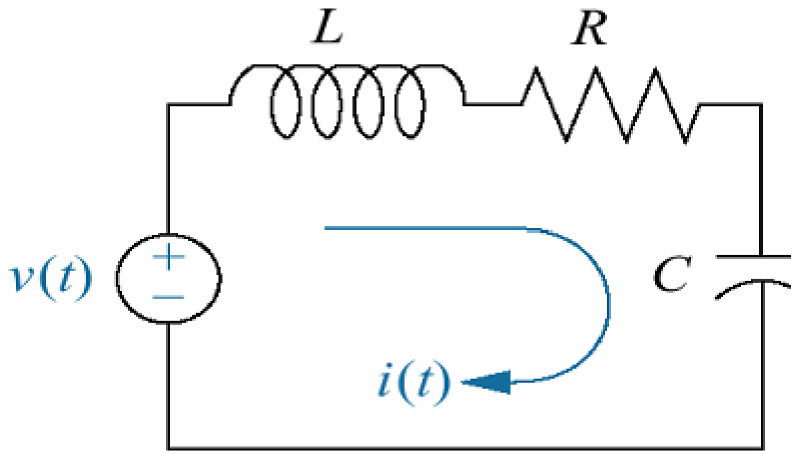
$$\frac{di}{dt} = \frac{1}{L}[v(t) - Ri(t)]$$

Thus, knowing the state variable, i(t), and the input v(t), we can find the value, or state, of any network variable at any time, $t \geq t_0$. Hence the algebraic equations of V_R and V_L are the *output equations.*

5.) Combining state equation and output equation is called *state-space represantation*.

Note : State variable are not unique. We could select $V_R(t)$ as state varible.

Let us now extend our observations to a second-order system and find the state-space represantation of this second-order system



1.) Since the network is second order, two simultaneous, first-order differential equation are needed to solve for two state variables. Select i(t) and q(t), the charge on the capacitor, as the two state variables.

2.) Write the loop equation $L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt=v(t)$

Converting the charge, using i(t)=dq/dt, we get $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=v(t)$

But an *n*thorder differential equaiton can be converted to *n* simultaneous first-order differential equation of the form $dx_i/dt = a_{i1}x_1+a_{i2}x_2+\ldots..+a_{in}x_n+b_i f(t)$ where each a_{ij}'s and b_i are constants for linear time-invariant systems. We say that right-hand side of this equation is a *linear combination* of the state variables and the input, f(t). That's why, we can convert the loop equation into two simultaneous first-order differential equations in terms of i(t) and q(t). The first equation can be dq/dt=i. The second equation can be formed by substituting ∫id(t)=q into the first equation and solving for di/dt. Summarizing the two resulting equations, we get

$$\frac{dq}{dt}=i$$

$$\frac{di}{dt}=-\frac{1}{LC}q-\frac{R}{L}i+\frac{1}{L}v(t)$$

3.) These equations are the state equations.

4.) From these two state variables, we can solve for all other network variables. For example, the voltage across the inductor can be written in terms of the solved state variables and the input as

$$V_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

This equation is an output equation. We say that $V_L(t)$ is a *linear combination* of the state variables, q(t) and i(t), and the input, v(t).

5.) The combined state equation and output equation is called as *state-space representation.*

Another choice of two state variables can be made. For example, $V_R(t)$ and $V_C(t)$.

<u>Is there any restriction on the choice of state variables</u>? **YES!** The restriction is that no state variable can be chosen if it can be expressed as a linear combination of the other state variables. For example, if $V_R(t)$ is chosen as a state variable, then i(t) can not be chosen, because $V_R(t)$ can be written as a linear combination of i(t), namely

$$V_R(t)=Ri(t)$$

Under these circumstances we say that the state variables are ***linearly independent***. State variables must be ***linearly independent***; that is, no state variable can be written as a linear combination of all pther state variables.

The state and output equations can be written in vector-matrix form if the system is linear. Thus, the state equations in this example can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\dot{x} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \qquad x = \begin{bmatrix} q \\ i \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \qquad u = v(t)$$

The output equation can be written as

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathrm{D}u$$

where

$$y = v_L(t) \qquad C = \begin{bmatrix} -1/C & -R \end{bmatrix} \qquad x = \begin{bmatrix} q \\ i \end{bmatrix} \qquad D = 1 \qquad u = v(t)$$

We call the combination of the equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ *and* $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathrm{D}u$ a *state-space representation* of the RLC network given in the last figure. A state-space representation, therefore, consist of (1) the simultaneous first-order differential equations and (2) the output equation. The representation developed in this section were for single-input, single-output system. For multi-input, multi-output case, y and u become vector quantities, and D become a matrix. Now, we will generalize the representation for multiple input, multiple output systems and summarize the concept of state-space representation.

THE GENERAL STATE-SPACE REPRESENTATION

Some Concepts and Terminology

Linear Combination : A linear combination of n variables, x_i, for i=1 to n, is given by the following sum, S; $S=K_n x_n+K_{n-1}x_{n-1}+\ldots\ldots+k_1x_1$ where each K_i is a constant.

Linear independence : A set of variables is said to be linearly independent if none of the variables can not be written as a linear combination of the others. For example, given x_1, x_2 and x_3, if $x_2=3x_3+2x_1$, then variables are not linearly independent.

System variable: Any variable that responds to an input or initial conditions in a system.

State variables : The smallest set of linearly independent system variables.

State vector : A vector whose elements are the state variables.

State space : The *n*-dimensional space whose axis are the state variables.

State equations : A set of *n* simultaneous first order differential equation with *n* variables.

Output equation : The algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

Now that the definitions have been formally stated, we define the state-space representation of a system.

A system is represented in state-space by the following equations

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}}$$

for $t \geq t_0$ and the initial conditions, $x(t_0)$, where

x = state vector

$\dot{x}$ = derivative of the state vector with respect to time

y = output vector

u = input or control vector

A = system matrix

B = input matrix

C = output matrix

D = feedforward matrix

The first equation is called ***state equation***, and the vector x, the ***state vector***, contains the state variables. The second equation is called ***output equation***. This representation of a system provides complete knowledge of all variables of the system at any time $t \geq t_0$.

The first step in representing a system is to select the state vector, which must be chosen according the following considerations:

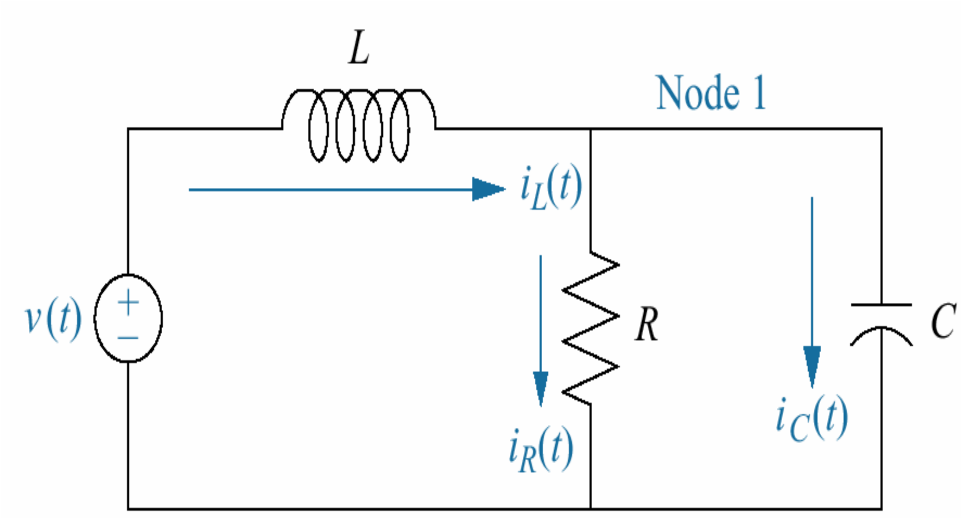
1.) A minimum number of state variables must be selected as components of the state vector.

2.)The components of the state vector (that is, this minimum number of state variables) must be linearly independent.

How do we know the minimum number of state variables to select? Typically, the minimum number required equals to the order of differential equaiton describing the system. Another way to determine the number of state variables is to count the number of independent energy storage elements in the system.

The following example demonstrates one technique for selecting state variables and representing a system in state-space. Our approach is to write the simple derivative equation for each derivative term as a linear combination of any of the system variables and the input that are present in the equation. Next we select each differentiated variable as a state variable. Then we exspress allother state variables in the equations in terms of the state variables and the input. Finally, we write the output variables as linear combinations of the state variables and the input.

Example : Given the electrical network of figure below, find a state-space representation if the output is the current through the resistor.



Solution : The following steps will yield a viable representation of the network in state-space.

Step 1 Label all of the branch currents in the network. These include i_L, i_R and i_C as shown in the figure.

Step 2 Select the state variables by writing the derivative equation for all energy-storage elements, that is, the inductor and capacitor. Thus, $C\frac{dv_C}{dt} = i_C$ and $L\frac{di_L}{dt} = V_L$

Using these two equations, choose the state variables as the quantities that are differentiated, namely v_C and i_L. We see that the state-space representation is complete if the right-hand side of these two equations can be written as linear combinations of the state variables and the input. Our next step is to express iC and vL as linear combinations of the state variables, v_C and i_L and the input v(t).

Step 3 Apply the network theory to obtain i_C and v_L in terms of the state variables v_C and i_L. At Node 1, $i_C = -i_R + i_L = -(1/R)v_C + i_L$ which yields i_C in terms of the state variables v_C and i_L. Around the outer loop, $v_L = -v_C + v(t)$ which yields v_L in terms of the state variable v_C and the source v(t).

Step 4 Using the loop equations we wrote in the previous step, obtain the following *state equations*:

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L \qquad L\frac{di}{dt} = -v_C + v(t)$$

or

$$\boxed{\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L \qquad \frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t)}$$

Step 5 Find the *output equation*. Since the output is $i_R(t)$,

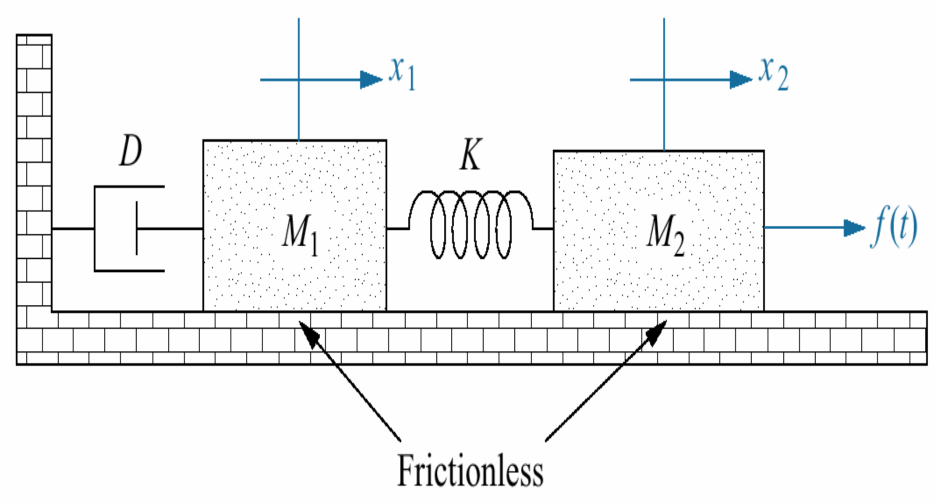
$$\boxed{i_R = \frac{1}{R}v_C}$$

The final result for the state–space representation in vector-matrix form is

$$\boxed{\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)}$$

$$\boxed{i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}}$$

Example : Find the state equations for the translational mechanical system shown in the figure below.



Solution : Write the differential equation for the network using the methods in Lecture 2 to find Laplace transformed equations of the motion. Next take the inverse Laplace transform of these equation assuming zero initial conditions

$$M_1 \frac{d^2 x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2 \frac{d^2 x_1}{dt^2} + Kx_2 = f(t)$$

Now let $d^2x_1/dt^2 = dv_1/dt$ and $d^2x_2/dt^2 = dv_2/dt$ and then select x_1, v_1, x_2 and v_2 state variables. Next form two of the state equations by solving the first equation above for dv_1/dt and for the second equation for dv_2/dt. Finally, add $dx_1/dt=v_1$ and $dx_2/dt=v_2$ to complete the set of state equations. Hence,

$$\frac{dx_1}{dt} = v_1 \qquad \frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2 \qquad \frac{dx_2}{dt} = v_2 \qquad \frac{dv_2}{dt} = \frac{K}{M_2}x_1 - \frac{K}{M_2}x_1 + \frac{1}{M_2}f(t)$$

In vector-matrix form;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

What is the output equation if the output is $v_0(t)$?

CONVERTING A TRANSFER FUNCTION TO STATE-SPACE

We will learn how to convert a transfer funtion represantation to a state-space representation. Let us begin by showing how to represent a general *n*th-order, linear differential equation with constant coefficients in state-space in the phase variable-form. We will then show how to apply this representation to transfer function.

Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + ...+ a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

A convenient way to choose state variables is to choose the output, y(t), and its (n-1) derivativesas the state variables. This choice is called ***phase-variables choice***. Choosing the state variables, x_i, we get

$$x_1 = y \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2 y}{dt^2} \qquad \cdot \qquad \cdot \qquad \cdot \qquad x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

Differetiating both sides yields →

$$\dot{x}_1 = \frac{dy}{dt} \qquad \dot{x}_2 = \frac{d^2 y}{dt^2} \qquad \cdot \qquad \cdot \qquad \cdot \qquad \dot{x}_n = \frac{d^n y}{dt^n}$$

Note that →

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \cdot \qquad \cdot \qquad \cdot \qquad \dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - .... - a_{n-1} x_n + b_0 u$$

In vector matrix form

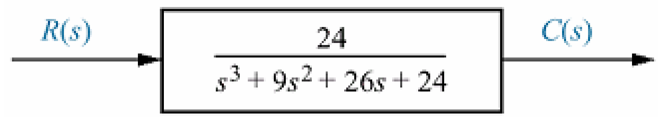
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0. & . & . & . & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & . & . & . & 0. \\ . & & & & & & & & & . \\ . & & & & & & & & & . \\ . & & & & & & & & & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & . & . & . & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ b_0 \end{bmatrix} u$$

This form is the phase-variable form of the state equations. Finally, the solution to the differential equation is y(t), or x_1, the output equation is

$$y = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}$$

In summary, to convert a transfer function into state equations in phase-variable form, we first convert the transfer function to a differential equation by cross-multiplying and taking the inverse Laplace transform, assuming zero initial conditions. Then we represent the differential equation in state-space in phase-variable form. An example illustrates the process.

Example : Find the state-space representation in phase-variable form for the transfer funtion shown in the figure below.



Solution : **Step 1** find the associated differential equation. Since $\dfrac{C(s)}{R(s)} = \dfrac{24}{(s^3+9s^2+26s+24)}$

Cross-multiplying yields $(s^3+9s^2+26s+24)C(s) = 24R(s)$. The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Step 2 Select the state variables. Choosing the state variables as succesive derivatives, we get

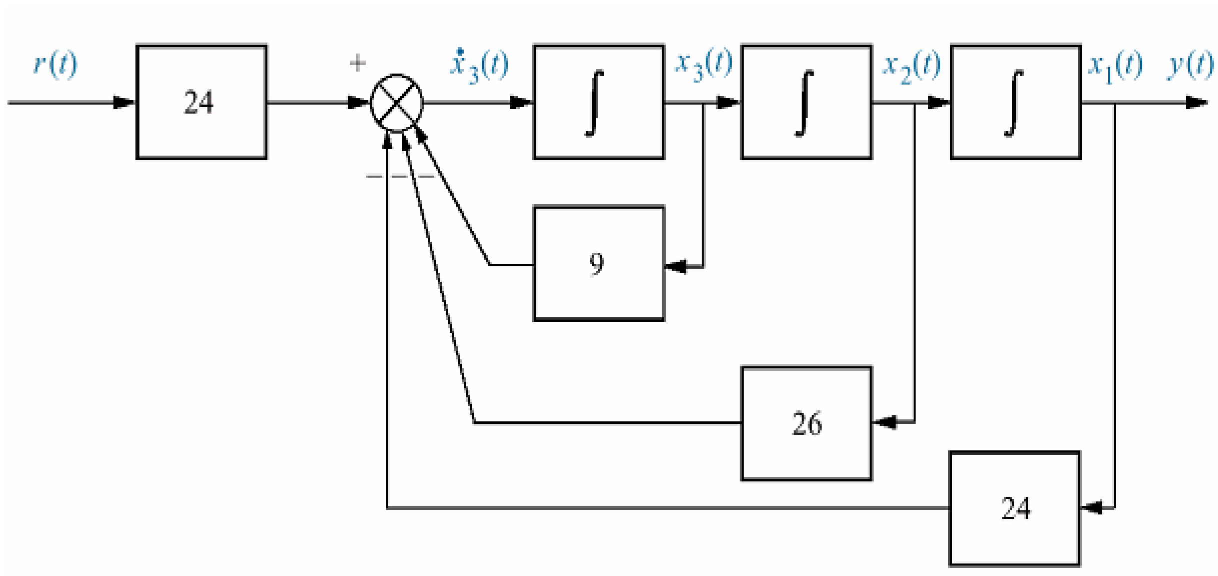
$$\begin{aligned} x_1 &= c \\ x_2 &= \dot{c} \\ x_3 &= \ddot{c} \end{aligned} \quad \xrightarrow{\text{Differentiating both sides}} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + 24r \end{aligned}$$

Clearly, output equation is $y = c = x_1$. In vector matrix form;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In this point we can create an equivalent block diagram of the system to help visualize the ssate variables. We draw three integral blocks as shown in figure below and label each output as one of the state variables.



CONVERTING FROM STATE-SPACE TO A TRANSFER FUNCTION

Given the state and output equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

take the Laplace transform assuming zero initial conditions:

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

solving for X(s) yields

$$(sI - A)X(s) = BU(s) \quad \text{or} \quad X(s) = (sI - A)^{-1} BU(s)$$

where I is identity matrix. Substituting the equation $X(s) = (sI - A)^{-1} BU(s)$ into equation y=Cx+Du yields

$$Y(s) = C(sI-A)^{-1}BU(s)+DU(s)$$

$$= [C(sI-A)^{-1}B+D]U(s)$$

We call the matrix $[C(sI-A)^{-1}B+D]$ the transfer function matrix, since it relates the output vector, Y(s), to the input vector, U(s). Hovewer, if U(s) and Y(s) are scalars, we can find the transfer function

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

Let us look at an example.

Example : Given the system defined by the following equations, find the transfer function T(s)=Y(s)/U(s), where U(s) is the input and Y(s) is the output.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Solution : First find (sI-A),

$$(sI-A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

Now form $(sI-A)^{-1}$:

$$(sI-A)^{-1} = \frac{adj(sI-A)}{\det(sI-A)} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2 2s+1}$$

Substituting $(sI-A)^{-1}$, B, C and D into equation $T(s) = \dfrac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$ we obtain the final result transfer function :

$$\boxed{T(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}}$$

LINEARIZATION

A prime advantage of the state-space representation over the transfer function representation is the ability to represent systems with nonlinearities. The ability to represent nonlinear systems does not imply the ability to solve their state equations for state variables and output.

If we are interested in small perturbations about an equilibrium point, we can also linearize the state equations about the equilibrium point. The key to linearization an about equilibrium point is, once again, the Taylor series. In the following example we write the state equations for a simple pendulum, showing that we can represent a nonlinear system in state-space; then we linearize the pendulum about its equilibrium point, the vertical position with zero velocity.

Example : First represent the simple pendulum shown in the figure in state space: *Mg* is the weight, T is an applied torque in the θ direction, and L is the length of the pendulum. Assume mass is evenly distributed, with the center of mass at L/2. Then linearize the state equations about the pendulum's equilibrium point – the vertical position with zero angular velocity.

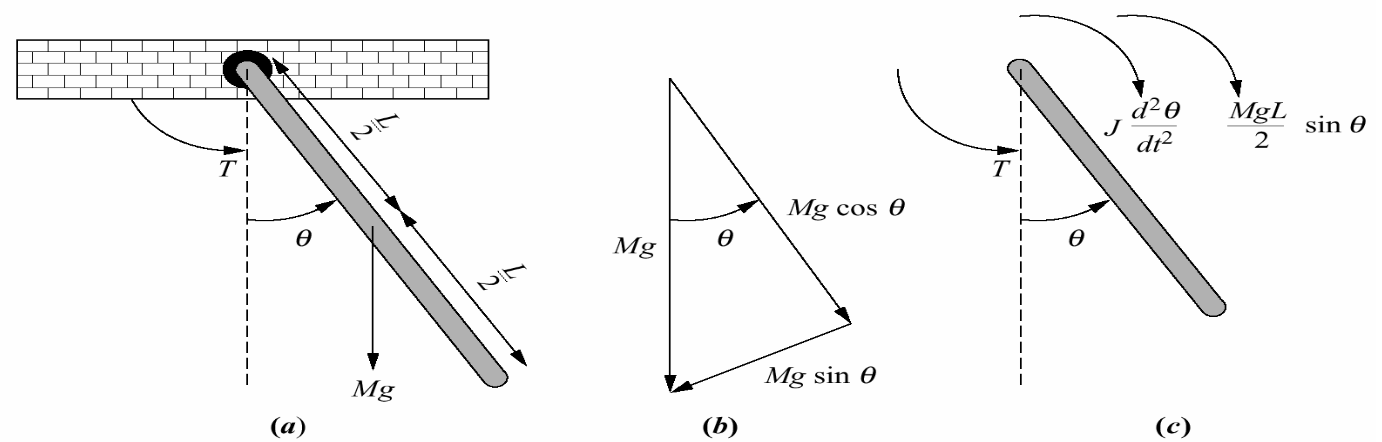


Figure 3.14
a. Simple pendulum;
b. force components of Mg;
c. free-body diagram

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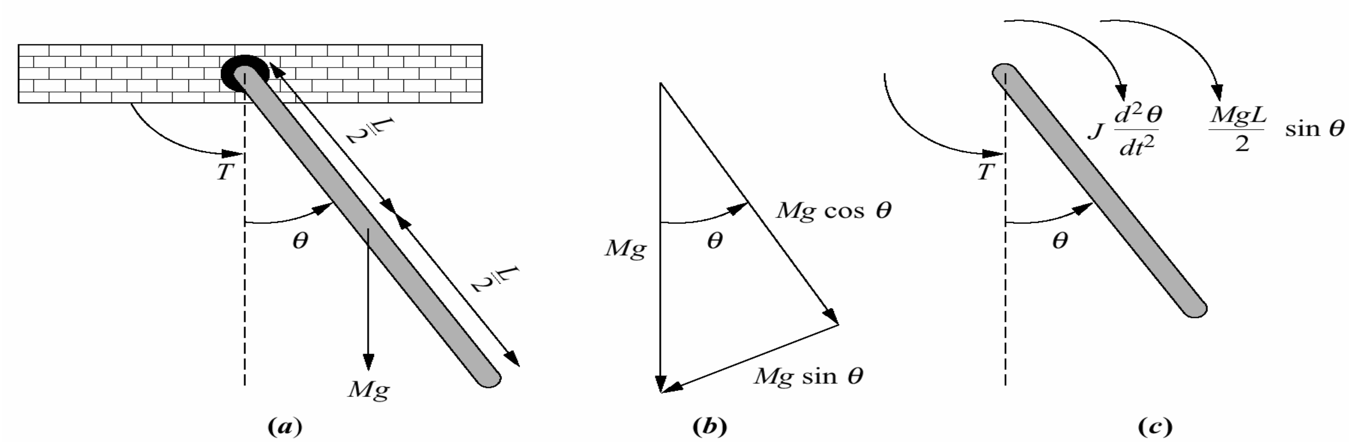

First we draw a free-body diagram as shown in Figure(c). Summing the torques, we get

$$J\frac{d^2\theta}{dt^2}+\frac{MgL}{2}\sin\theta=T$$

where J is the moment of inertia of the pendulum around the point of rotation. Select the state variables, x_1 and x_2 as phase variables. Letting $x_1=\theta$ and $x_2=d\theta/dt$, we write the state equations as

$$\dot{x}_1=x_2$$

$$\dot{x}_2=-\frac{MgL}{2J}\sin x_1+\frac{T}{J}$$

Thus we have represented a nonlinear system in state-space. If we want to apply classical techniques and convert these state equation to transfer function, we must linearize them. Let us proceed now to linearize the equation about the equilibrium point, $x_1=0$, $x_2=0$, that is, $\theta=0$ and $d\theta/dt=0$.

Let x_1 and x_2 be perturbed about the equilibrium point, or

$$x_1 = 0 + \delta x_1$$

$$x_2 = 0 + \delta x_2$$

Using the linearizing equation we have seen formerly $f(x) - f(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$

we obtain $\sin x_1 - \sin 0 = \left.\frac{d(\sin x_1)}{dx_1}\right|_{x_1=0} \delta x_1 = \delta x_1$ from which $\sin x_1 = \delta x_1$

Substituting the last two equations into the state equation which is $\dot{x}_2 = -\frac{MgL}{2J}\sin x_1 + \frac{T}{J}$

yields the following state equations

$$\dot{\delta} x_1 = \delta x_2$$

$$\dot{\delta} x_2 = -\frac{MgL}{2J}\delta x_1 + \frac{T}{J}$$

which are linear and a good approximation to original state equations for small excursions away from the equilibrium point.

What is the output equation ?