TIME RESPONSE

In this chapter you will learn the following:

❖How to find time response from the transfer function

❖How to use poles and zeros to determine the response of a control system

❖How to describe the transient response of first- and second-order systems

❖How to view the effects of nonlinearities on the system time response

❖How to find time response from the state-space representation

This lecture is devoted to the analysis of system transient response. First we must describe some valuable analysis and design tools, poles, zeros and system response.

The **_poles_** of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become infinite or (2) any roots of the denominator of the transfer function that are common roots of the numerator.
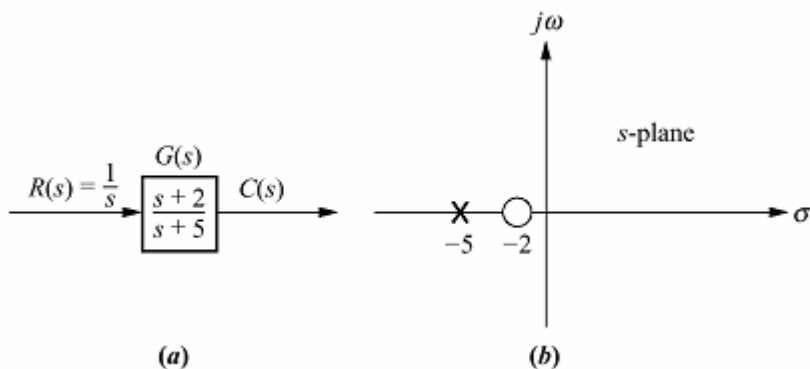
The **_zeros_** of a transfer funtion are (1) the values of the Laplace transform variable, s, that cause the transfer function become zero, or (2) any roots of the numerator of the transfer function that are common roots of the denominator.
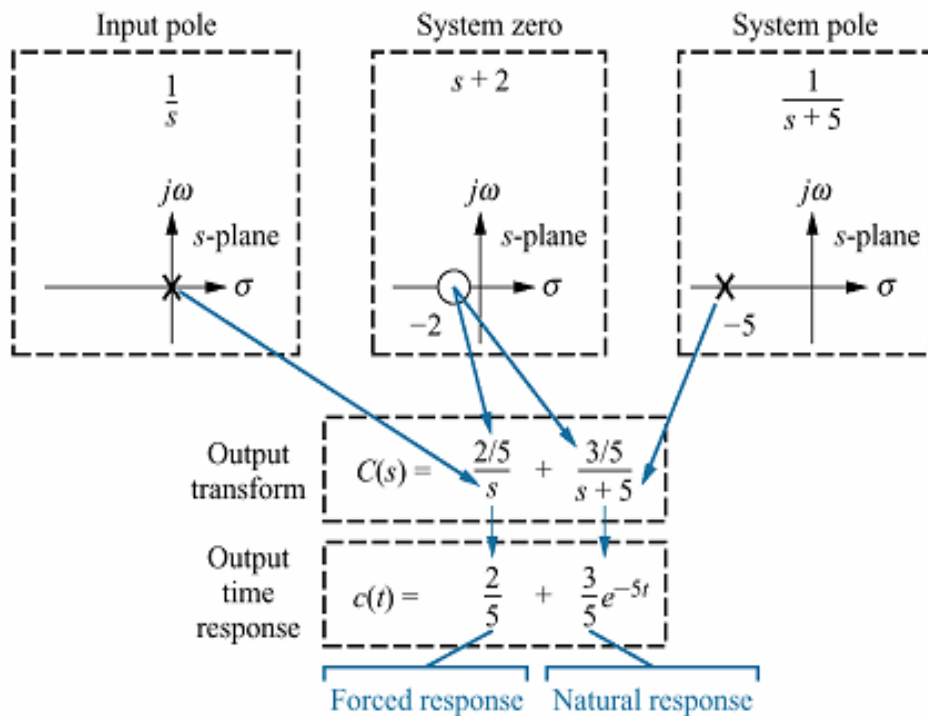
POLES AND ZEROS OF A FIRST ORDER SYSTEM : AN EXAMPLE

Given the transfer function G(s) in Figure (a), a pole exists at s=-5 and a zero exists at -2. these values are plotted on the complex as plane in Figure(b) using an x for the pole and a o for the zero. Two show the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function of Figure(a) by a step function yields

$$C(s) = \frac{s+2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

where

$$A = \frac{s+2}{s}\bigg|_{s\to 0} = \frac{2}{5}$$

and

$$B = \frac{s+2}{s}\bigg|_{s\to -5} = \frac{3}{5}$$

Thus,

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

All of these are summarized in the follwing figure(c) and we draw the following conclusions :

(a)

(b)

Input pole

$\frac{1}{s}$

$j\omega$

s-plane

σ

System zero

$s+2$

$j\omega$

s-plane

σ

-2

System pole

$\frac{1}{s+5}$

$j\omega$

s-plane

σ

-5

Output transform

$C(s) = \dfrac{2/5}{s} + \dfrac{3/5}{s+5}$

Output time response

$c(t) = \dfrac{2}{5} + \dfrac{3}{5}e^{-5t}$

Forced response   Natural response

Conclusions :

1.) A pole of the input function generates the form of the *forced response* (that is, the pole at the origin generated a step function at the output).

2.) A pole of the transfer function generates the form of the *natural response* (that is, the pole at -5 generated e^{-5t}).

3.) A pole on the real axis generates an *exponential* response of the form e$^{-\alpha t}$

where –α is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero (again, the pole at -5 generated e^{-5t})

4.) The zeros and poles generate the *amplitudes* for both the forced and natural response

Let us look at an examplethat demonstrates the technique of using poles to obtain the form of the system response

Example : A system has a transfer function $G(s) = \dfrac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$

Write, by inspection, the output, c(t), in general terms if the input is a unit step.

Answer : $\qquad$ c(t) = A + Be^{-t} + Ce^{-7t} + De^{-8t} + E^{-10t}

By now, we learned that poles determine the nature of the time response: Poles of the input function determine the form of the forced response, and poles of the transfer function determine the the form of the natural response. Zeros and poles of input or transfer function contribute to the amplitudes of the components parts of the total response. Finally, poles on yhe real axis generate exponentials responses.

FIRST-ORDER SYSTEMS

We now discuss first-order systems without zeros to define a performance specification for such a system. A first-order without zeros can be described by the transfer funtion given in the figure(a). If the input is a unit step, where R(s) = 1/s, the Laplace transform of the step response is C(s), where $C(s) = R(s)G(s) = \dfrac{a}{s(s+a)}$. Taking the inverse Laplace

transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

where the input pole at the origin generated the forced response c$_f$(t)=1, and the system pole at –a, as shown in Figure(b).

(a)

(b)
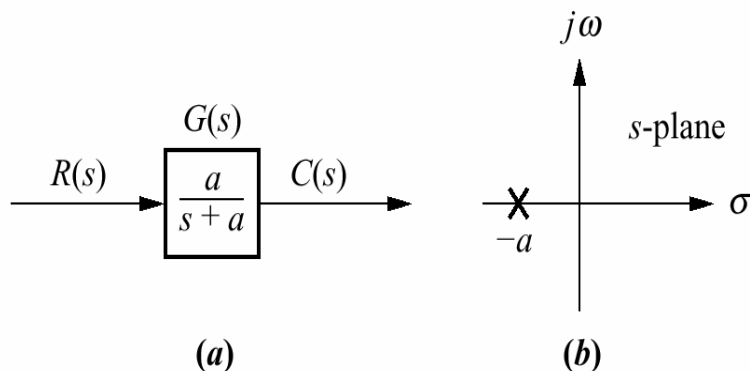
$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \quad \textbf{(*)}$$

Let us examine the significance of parameter a, the only parameter needed to describe the transient response. When t=1/a,

$$e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37 \quad \textbf{(**)} \qquad \text{or} \qquad c(t)\Big|_{t=1/a} = 1 - e^{-at}\Big|_{t=1/a} = 1 - 0.37 = 0.63 \quad \textbf{(***)}$$

We now use these equations to define three **_transient response specifications_**.

Time Constant : We call 1/a the **time constant** of the response. From equation (*), the time constant can be described as the time for e^{-at} to decay to 37% of its initial time. Alternately, from equation (**), the time constant is the time it takes for the step response to rise to 63% of its final value. Thus, we can call the parameter *a* as *exponential frequency*.
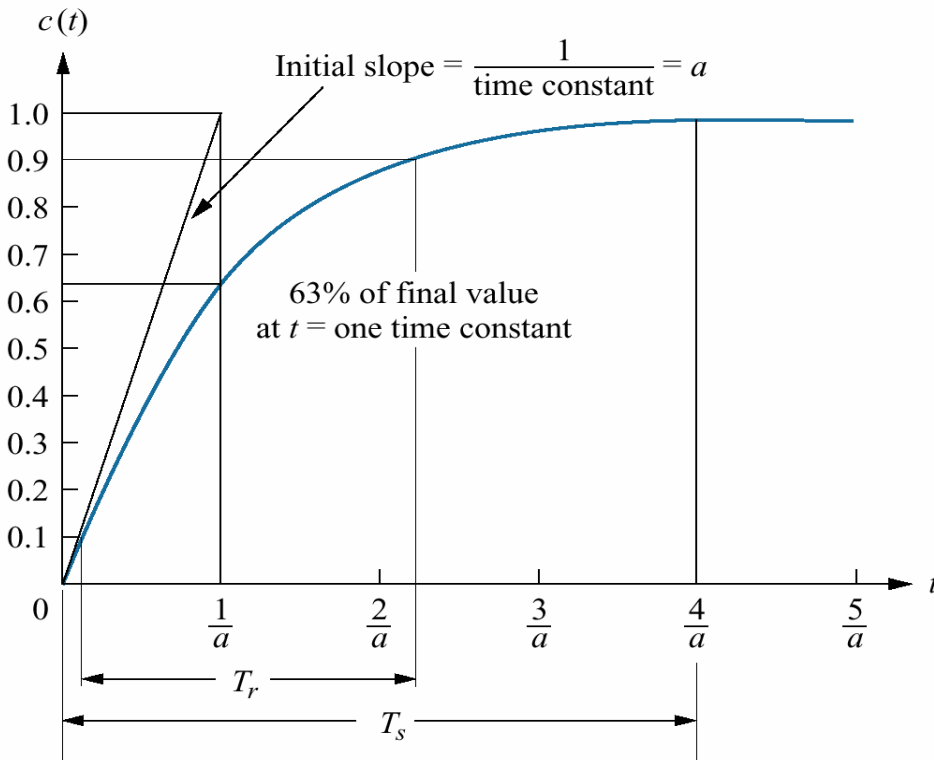
Rise Time, T_r : Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value. Rise time is found by solving equation (*) and solving for time at c(t)=0.9 and c(t)=0.1. Hence
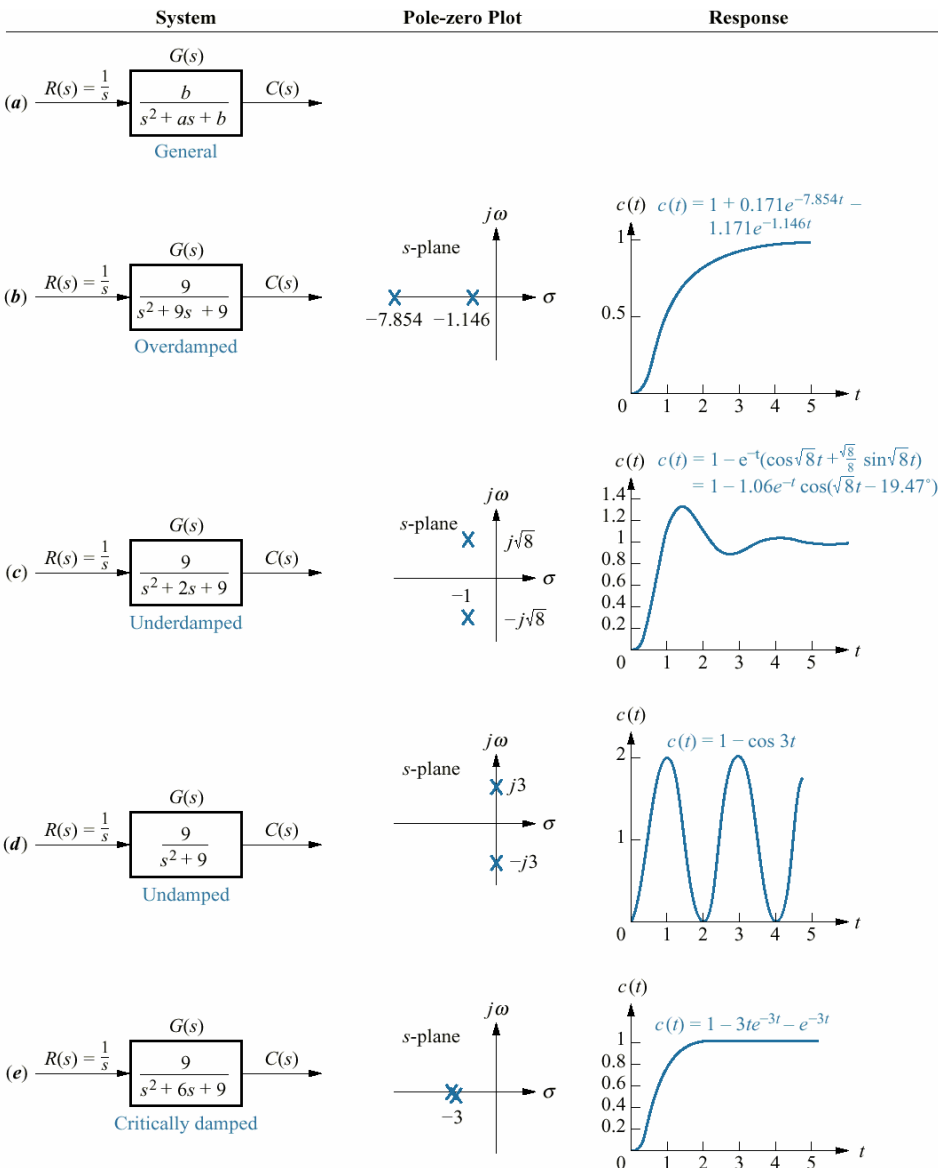
$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

Settling Time, T_s : Settling time is defined as the time for the response to reach and stay within 2% of its final value. Letting c(t)=0.98 in equation (*) and solving for t, we find the settling time to be

$$T_s = \frac{4}{a}$$

SECOND-ORDER SYSTEMS : INTRODUCTION

Let us now extend the concepts of poles, zeros and transient response to second order systems. Compared to the simplicity of a first ordersystem, a second order system exhibits a wide range of responses that must be analyzed and described. To become familiar with the wide range of responses before formalizing our discussion, we take a look at numerical examples of the second order system responses shown in the figure. All examples are derived from Figure(a), the general case, which has two finite poles and no zeros. By assigning appropriate values to parameters a and b, we can show all possible second,order transient responses. We now explain each response and show how we can use the poles to determine the nature of the response without going through the procedure of a partial-fraction expansion followed by the inverse Laplace transform.

System	Pole-zero Plot	Response

(a) $R(s) = \frac{1}{s}$ → $G(s)$ $\dfrac{b}{s^2 + as + b}$ → $C(s)$

General

(b) $R(s) = \frac{1}{s}$ → $G(s)$ $\dfrac{9}{s^2 + 9s + 9}$ → $C(s)$

Overdamped

s-plane: $j\omega$, poles at $\sigma = -7.854$ and -1.146

$c(t)$ $c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$

(c) $R(s) = \frac{1}{s}$ → $G(s)$ $\dfrac{9}{s^2 + 2s + 9}$ → $C(s)$

Underdamped

s-plane: poles at $\sigma = -1$, $j\sqrt{8}$ and $-j\sqrt{8}$

$c(t)$ $c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t)$
$= 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°)$

(d) $R(s) = \frac{1}{s}$ → $G(s)$ $\dfrac{9}{s^2 + 9}$ → $C(s)$

Undamped

s-plane: poles at $j3$ and $-j3$

$c(t)$ $c(t) = 1 - \cos 3t$

(e) $R(s) = \frac{1}{s}$ → $G(s)$ $\dfrac{9}{s^2 + 6s + 9}$ → $C(s)$

Critically damped

s-plane: poles at $\sigma = -3$

$c(t)$ $c(t) = 1 - 3te^{-3t} - e^{-3t}$

<u>Overdamped Response :</u> For this response,

$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s+7.854)(s+1.146)}$$
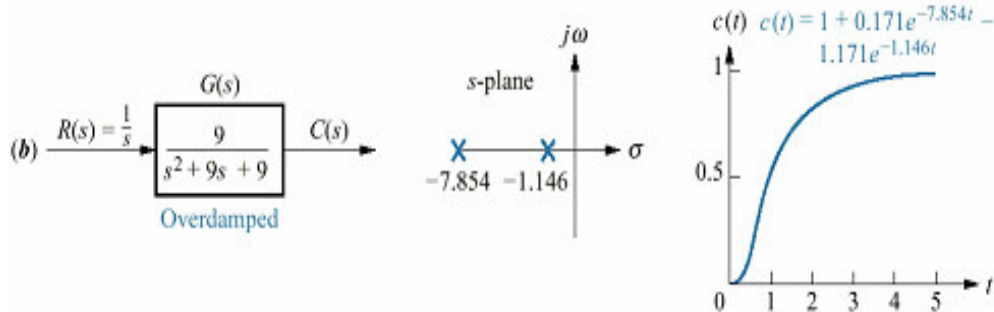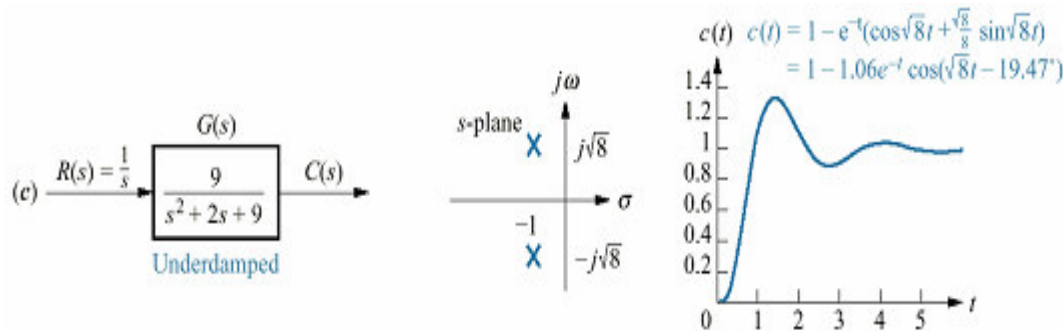

This function has a pole at the origin that comes from the unit step input and two real poles that come from the system. The input pole at the origin generates the constant forced response; each of two system poles on the real axis generates an exponential natural frequency. Hence the output could have been written as $c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$. This response shown in figure called **_overdamped_**. We see that the poles tell us the form of the response without the tedious calculation of the inverse Laplace transform.

**<u>Underdamped Response :</u>** For this response ,

$$C(s) = \frac{9}{s(s^2 + 2s + 9)}$$

This function has a pole at the origin that comes from the unit step and two complex poles that come from the system. The poles that generate the natural response are at $s = -1 \pm j\sqrt{8}$. The real part of the pole matches the exponential decay frequency os sinusoids's amplitude, while the imaginary part of the pole matches the frequecy of the snisoidal oscillation. The sinusoidal frequency is given the name of _damped frequency of oscillation, ω_d._ This response shown in figure called **_underdamped_**.
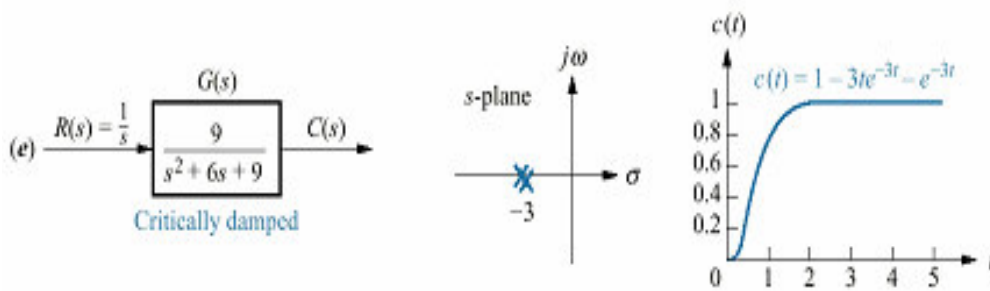
Underdamped Response : For this response, $C(s) = \dfrac{9}{s(s^2 + 9)}$
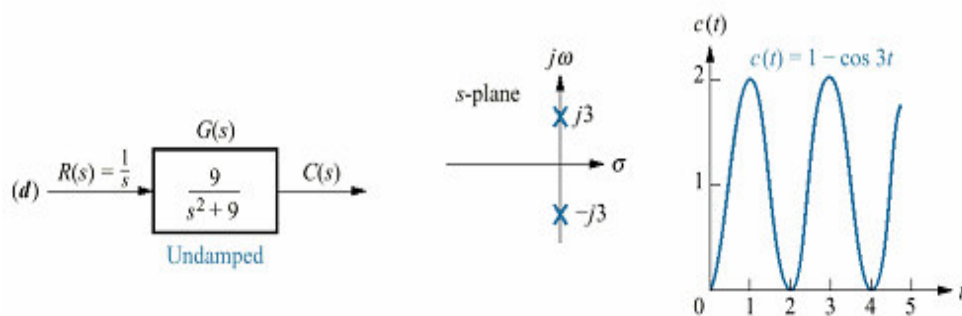
This function has a pole at the origin and two imaginary poles. The pole at the origin generates the constant forced response, and the two system poles on the imaginary axis at ±j3 generate a Sinusoidal natural response. Hence, the output can be estimated as $c(t) = K_1 + K_4\cos(3t-\varphi)$ This type of response is called undamped.

Critically Damped Response : For this response, $C(s) = \dfrac{9}{s(s^2 + 6s + 9)} = \dfrac{9}{s(s+3)^2}$

This function has a pole at the origin and two multiple real poles. The input pole at the origin generates the constant forced response, and two poles at the real axis at -3 generate a natural exponential response. Hence the output can be estimated as $c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$

We now summarize our observations. In this section we defined the following natural response and found their characteristics.

1. **_Overdamped responses :_** Poles : Two real at $-\sigma_1$, $-\sigma_2$

Natural reponse : Two exponentials with time constant equal to the reciprocal of the pole locations, or $\qquad c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$

2. **_Underdamped responses :_** Poles : Two complex at $-\sigma_d \pm j\omega_d$

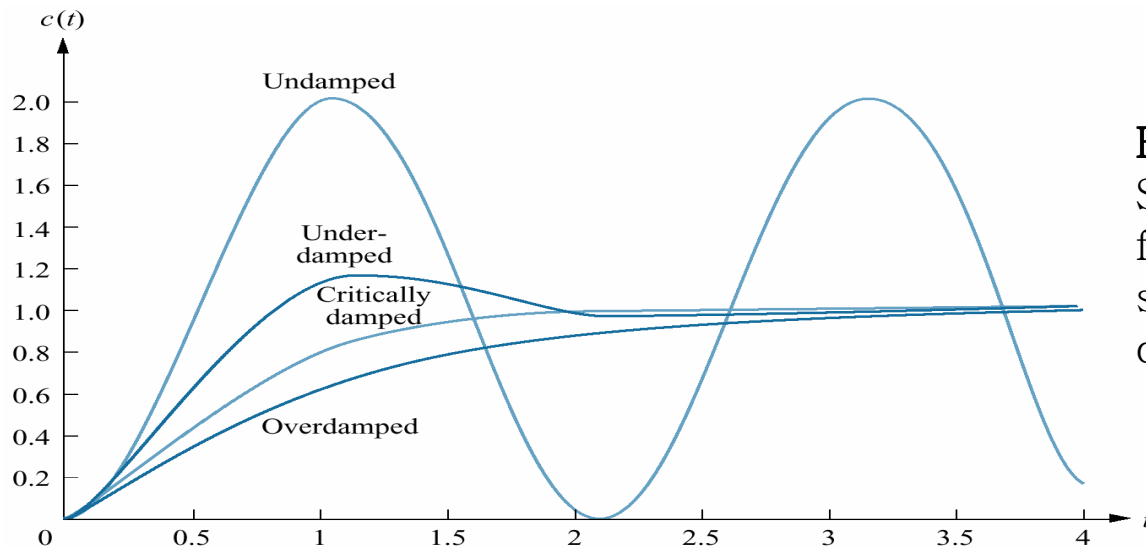Natural response : Damped snisoidal with an exponential envelope, or

$$c(t) = Ae^{-\sigma_d t} \cos(\omega_d t - \phi)$$

3. **_Undamped responses :_**   Poles : Two imaginary at $\pm j\omega_1$

Natural response : Undamped snisoid,   or $c(t) = A\cos(\omega_1 t - \phi)$

4. **_Critically damped responses :_** Poles : two real at $-\sigma_1$

Natural response : Two exponential term, or $c(t) = K_1 e^{-\sigma_1 t} + K_2 te^{-\sigma_1 t}$

Figure 4.10
Step responses
for second−order
system
damping cases

THE GENERAL SECOND-ORDER SYSTEM

In this section we will define two physically meaningful specificaitons for second-order system.

Natural Frequency, ω_n : The natural frequency of a second order system is the frequency of oscillation of the system without damping. For example, the frequency of oscillation of a series RLC circuit with the resistance shorted would be natural frequency.

Damping Ratio, ζ : We define the damping ratio, ζ, to be

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/sec)}} = \frac{1}{2\pi} \frac{\text{Natural period (second)}}{\text{Exponential time constant}}$$

Let us now revise our description of the second-order system to reflect the new definitions. A general second-order system can be transformed to show the quantities ζ and ω_n. Consider the general system

$$G(s) = \frac{b}{s^2 + as + b}$$

Without damping, the poles would be on the jω axis, and the response would be an undamped sinusoid. For the poles to be purely imaginary, a=0. Hence $G(s) = \frac{b}{s^2 + b}$

By definition, the natural frequency, ω_n, is the frequency of oscillation of this system. Since the poles of the system are on jω axis at $\pm j\omega\sqrt{b}$, $\omega_n = \sqrt{b}$ and $b = \omega_n^2$. Now, what is the value of a?

Assuming the underdamped system, the complex poles have a real part, σ, equal to -a/2. The magnitude of this value is then exponential decay frequency. Hence,

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency(rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}$$

from which $\boxed{a = 2\zeta\omega_n}$. Our general second order transfer function finally looks like this:

$$G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

In the following example we find numerical values of ζ and ω_n by matching the transfer function to final equation we found above

Example : Given the transfer function, find ζ and ω_n . $\qquad G(s) = \dfrac{36}{s^2 + 4.2s + 36}$

Solution : Note that $\omega_n^2 = 36$ and $\omega_n = 6$. Also, $2\zeta\omega_n = 42$ and $\zeta = 0.35$

We have defined ζ and ω_n , let us relate these quantities to the pole location. Solving for the poles of the transfer function in equation

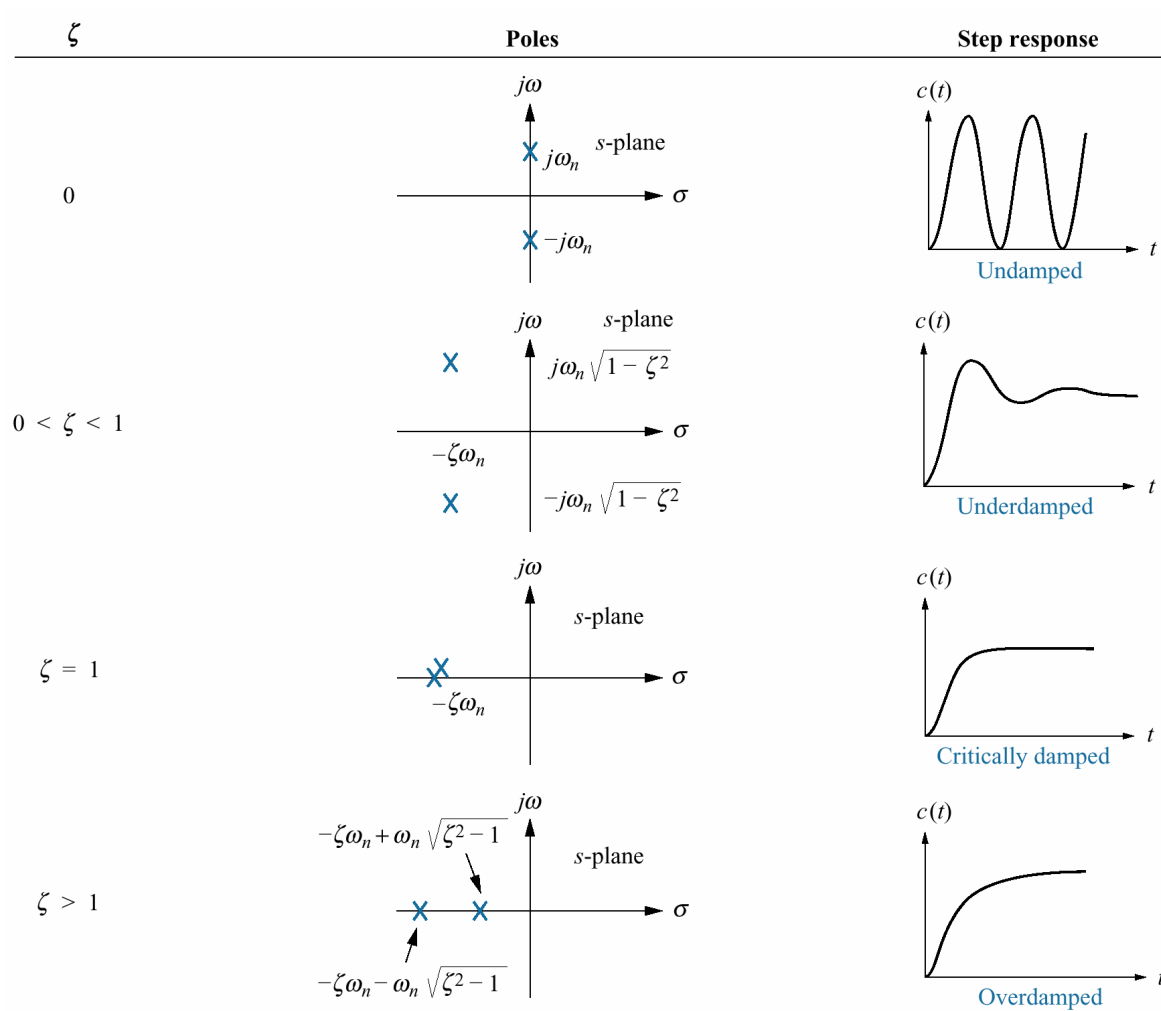$$G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

yields

$$s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

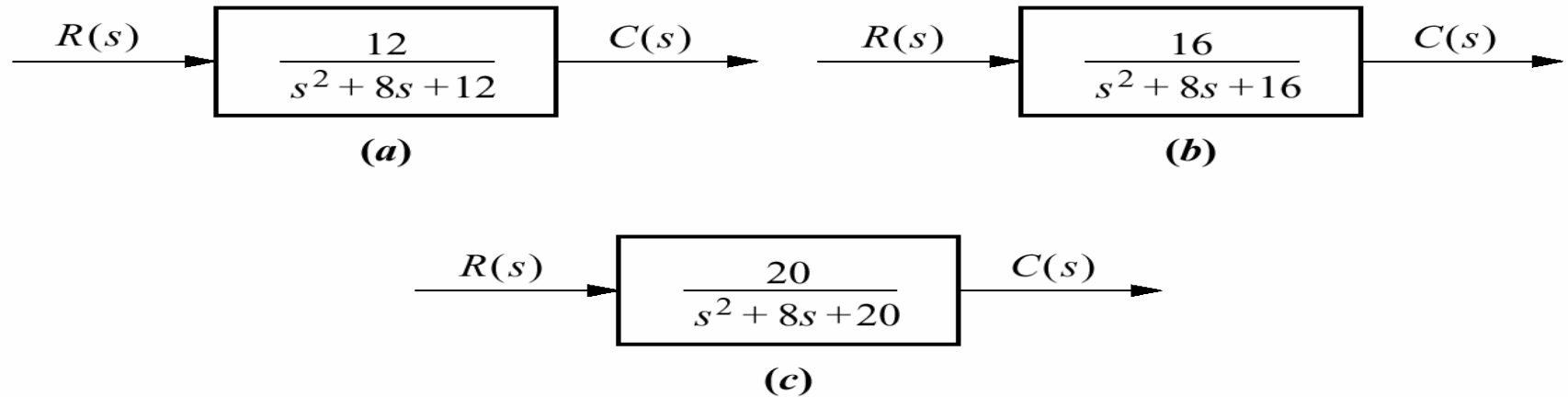$$s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

From the equation above, we see that the various cases of second-order response are a function of ζ ; they are summarized in the table below.

Figure 4.11
Second-order
response as a
function of
damping ratio

ζ	Poles	Step response

In the following example we will find the numerical value of ζ and determine the nature of transient response

Example : For each of the systems shown in the figure, find the value of ζ and report the kind of response expected.

$$R(s) \rightarrow \boxed{\dfrac{12}{s^2 + 8s + 12}} \rightarrow C(s)$$

(a)

$$R(s) \rightarrow \boxed{\dfrac{16}{s^2 + 8s + 16}} \rightarrow C(s)$$

(b)

$$R(s) \rightarrow \boxed{\dfrac{20}{s^2 + 8s + 20}} \rightarrow C(s)$$

(c)

Solution : First match the form of these systems to the forms shown in equations

$$G(s) = \frac{b}{s^2 + as + b} \quad \text{and} \quad G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Since a = $2\zeta\omega_n$ and $\omega_n=\sqrt{b} \rightarrow \zeta =a/(2\sqrt{b})$. Using the values of a and b from each of the systems of the figure, we find ζ=1.155 for system(a), which is overdamped, since ζ>1;

ζ =1 for system(b), which is thus critically damped; and ζ=0.894 for system(c), which is thus underdamped, since ζ<1.

Underdamped Second Order Systems

In addition to ζ and ω_n, more specifications indigenous to the underdamped case will be defined. These specifications are defined as follows:
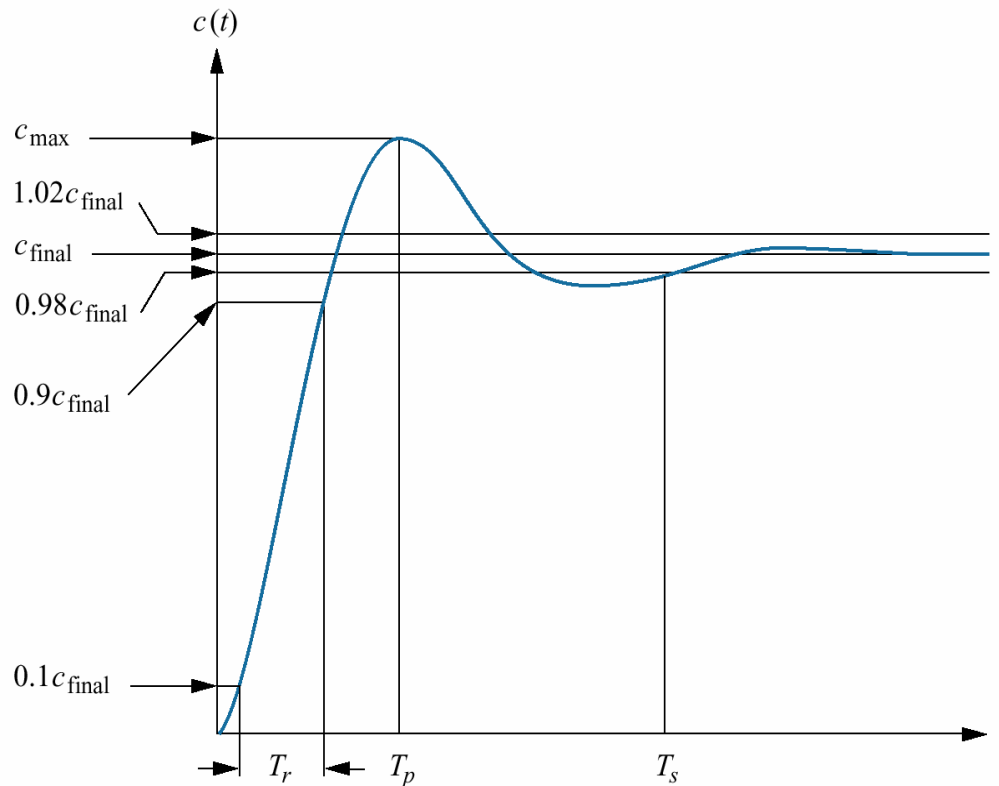

Figure 4.14
Second-order
underdamped
response
specifications

1. **Rise time**, T_r : the time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

2. **Peak time**, T_p : The time required to reach the first, or maximum, peak.

3. **Percent overshoot**, %OS : the amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

4. **Settling time**, T_s : The time required for the transient's damped ascillation to reach and stay within ±2% of the steady-state value.

Evaluation of T_p, %OS and T_s : To find the value of these parameters, we use the equations of

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

$$T_s = \frac{4}{\zeta\omega_n}$$

We can use the graphics in figure 4.16 to obtain the value of rise time, T_r .

Figure 4.16
Normalized rise time vs. damping ratio for a second-order underdamped response

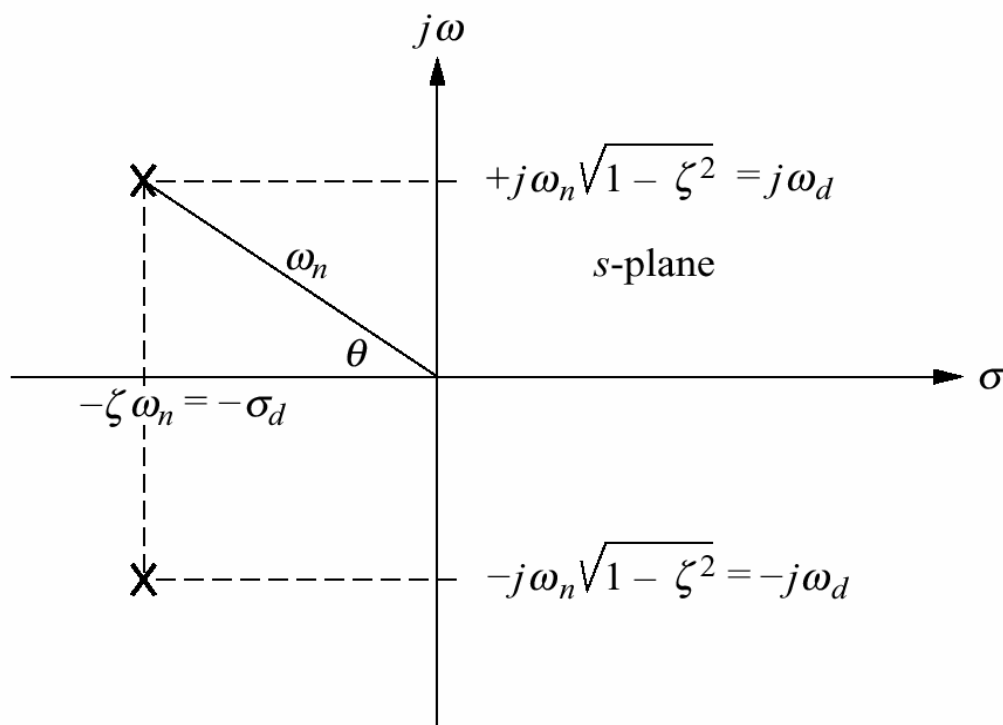| Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883
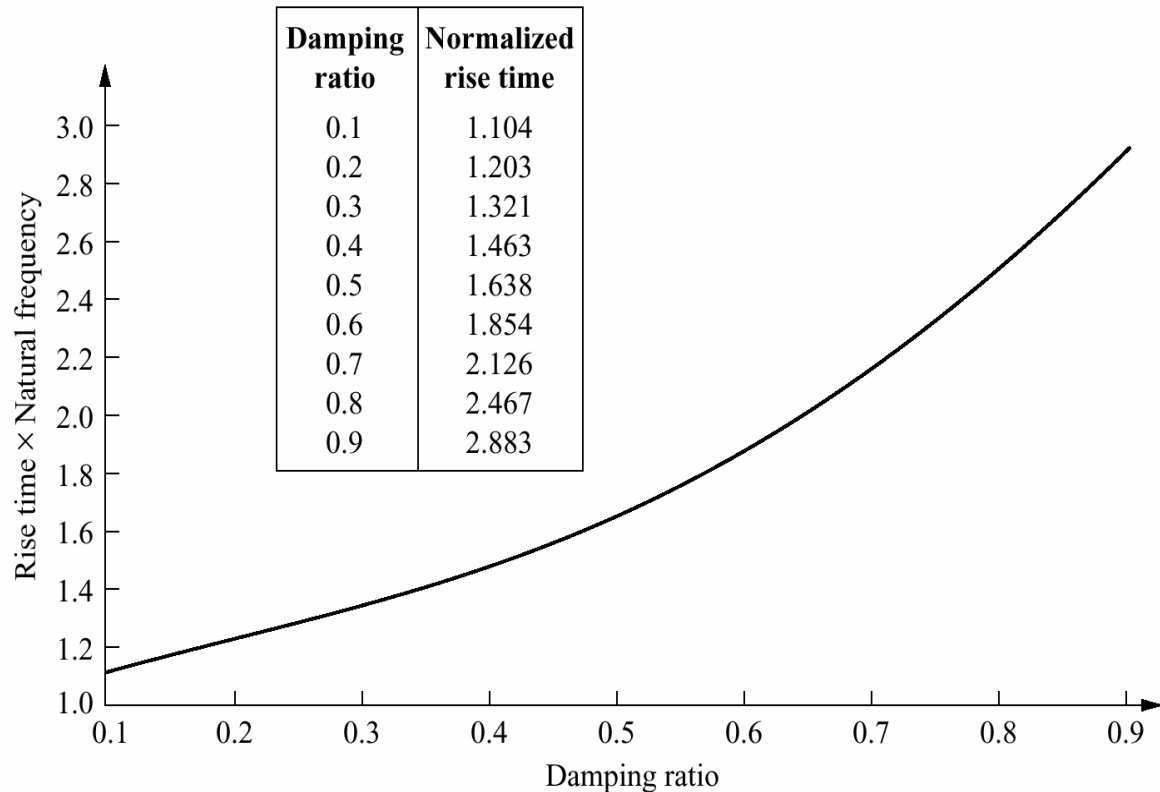
Let us look an example

Example : Given the transfer function $G(s) = \dfrac{100}{s^2 + 15s + 100}$, find T_p, %OS, T_s and T_r .

Solution : $\omega_n = 10$ and $\zeta = 0.74$. Using the related equation, $T_p = 0.475$ second, %OS=2.838, $T_s = 0.533$. Using the related graphic, normalised rise time is approximately 2.3 second. Dividing by ω_n yields $T_r = 0.23$ second.

We now have expressions that relate peak time, percent overshoot and settling time to the natural frequency nad the damping ratio. Now let us relate these quantities to the location of poles that generate these characteristics.

The pole plot for a general underdamped second-order system is in figure below. We see from the Pythagorean theorem that the radial distance from the origin to the pole is the natural frequency, ω_n, and the $\cos\theta$ = ζ. Now comparing the equations we have evaluated formerly with the pole location, we evaluate peak time and settling time in terms of pole location :
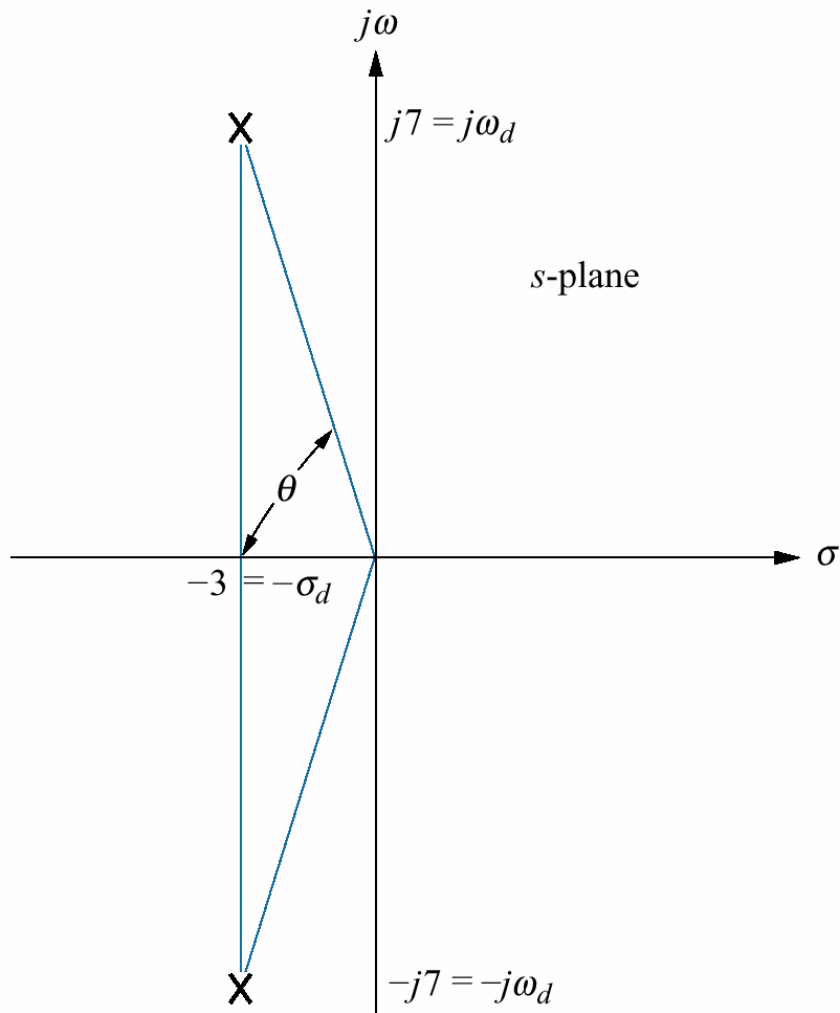

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

where ω_d is the imaginary part of the pole and is called the **damped frequency of oscillation**, and σ_d is the magnitude of the real part of pole and is **exponential damping frequency**

Example : Given the pole plot in figure, find ζ, ω_n, T_p, %OS and T_s .

Solution : The damping ratio is given by $\zeta=\cos\theta=\cos[\arctan(7/3)]=0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n=\sqrt{(7^2+3^2)}=7.616$. The peak time is

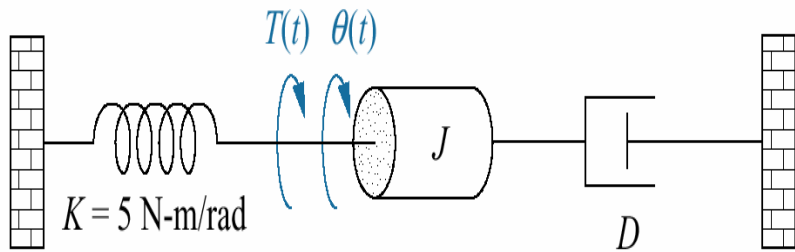$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \ \text{second}$$

The percent overshoot is

$$\% \, OS \ = \ e^{-\left(\zeta\pi \, / \, \sqrt{1-\zeta^2} \right)} \times 100 \ = \ 26 \ \%$$

The approximate settling time is

$$T_s \ = \ \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \ \text{second}$$

Example : Given the system shown in figure, find J and D to yield 20% overshoot and a settling time of 2 second for a step input of torque T(t).

$T(t)\ \theta(t)$

$K = 5$ N-m/rad

D

Solution : First, the transfer function for the system is

$$G(s) = \frac{1/J}{s^2 + \dfrac{D}{J}s + \dfrac{K}{J}}$$

From the transfer function, $\omega_n = \sqrt{\dfrac{K}{J}}$ and $2\zeta\omega_n = \dfrac{D}{J}$

But, from the problem statement, $T_s = 2 = \dfrac{4}{\zeta\omega_n}$ or $\zeta\omega_n = 2$. Hence $2\zeta\omega_n = 4 = \dfrac{D}{J}$.

Also, $\zeta = \dfrac{4}{2\omega_n} = 2\sqrt{\dfrac{J}{K}}$. We know the formulation for overshoot, which is $\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$

If we draw ζ from this equation, we find that a 20% overshoot implies $\zeta = 0.456$. Therefore

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} \Rightarrow \frac{J}{K} = 0.052$$

From the problem statement, K = 5 N-m/rad. Combining this value with equations

$2\zeta\omega_n = 4 = \dfrac{D}{J}$ and $\dfrac{J}{K} = 0.052$ which we have found above, we get

$$D = 1.04 \text{ N-m-s/rad} \quad \text{and} \quad J = 0.26 \text{kg-m}^2$$

SYSTEM RESPONSE WITH ADDITIONAL POLES AND ZEROS

By now, we analyzed systems with one or two poles and with one zeros. If a system has more than two poles or zeros we can not use the formulation we derived for calculation of performance specifications. But, under certain conditions, a system with more than two poles or zeros can be approximated as a second order system that has just two **dominant** poles. If the real pole is five times farther to the left than two complex-conjugate pole pair, we can say this complex-conjugate pole pair is dominant pole pair.

A way to look at the effect of an additional zerois as follows. Let C(s) be the response of a system, T(s), with unity in the numerator. If we add a zero to the transfer function, yielding (s+a)T(s), the Laplace transform of the response will be

$$(s+a)C(s) = sC(s) + aC(s)$$

Thus, a response of a system with a zero consists of two parts : The derivative of the original response and a scaled version of the original response. If *a* is negative, an interesting phenomenon occurs. The output may be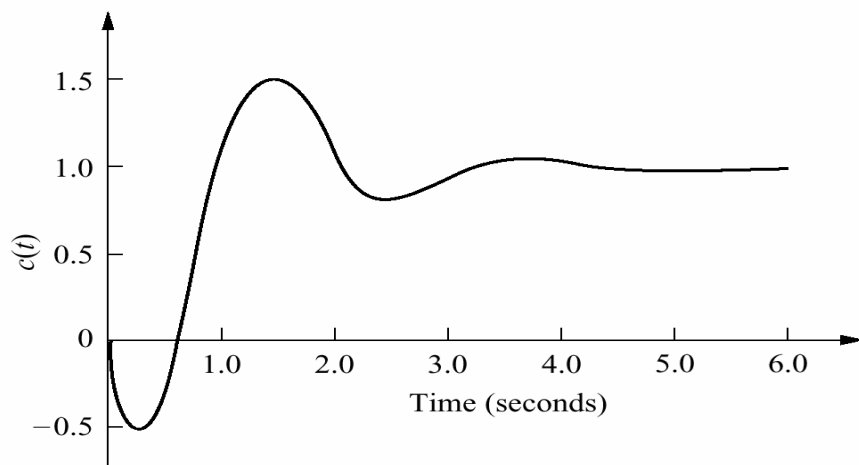 opposite sign of input as shown in figure. Along first 0.6 second, the output has negative sign although the input has positive sign (step input). A system that exhibits this phenomenon is known as a **nonminimum phase system**. If an airplane was a nonminimum phase system, it would initially veer left when commanded to steer right.

EFFECT OF NONLINEARITIES UPON TIME RESPONSE

In this section we will insert nonlinearities, such as saturation, dead zone and backlash into a system to show the effects of these nonlinearities upon the linear responses. The responses were obtained using Simulink.

Let us assume the motor and load from a Antenna Position Control System. The input of the system is applied voltage to motor, and the output of the system is load angular velocity. Let us add a saturation(±5 V) to system.
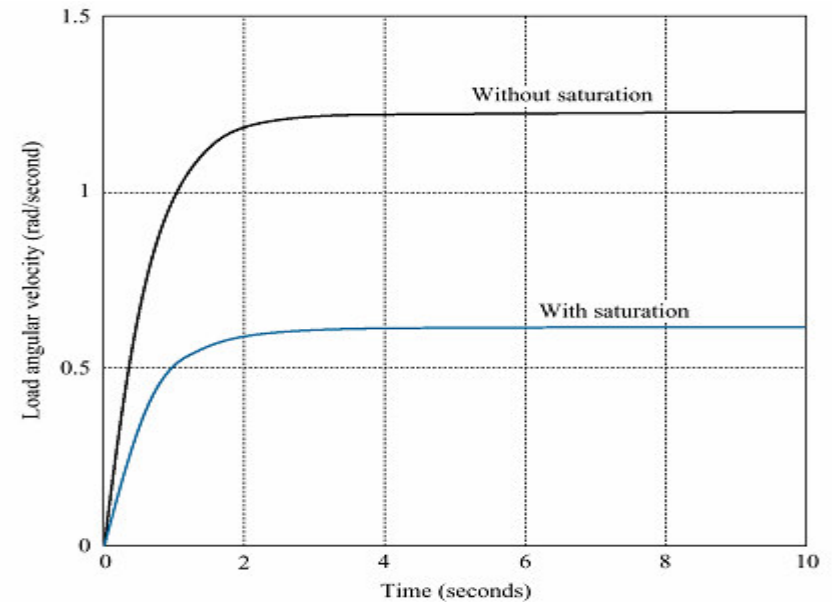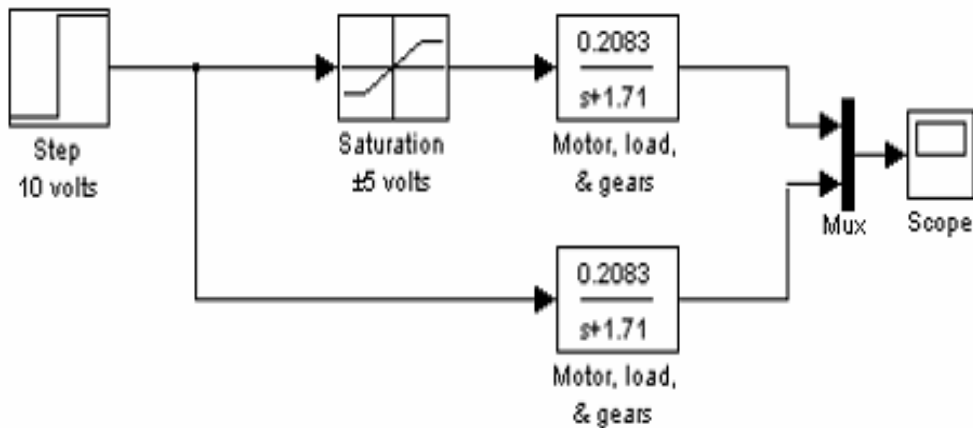
As shown in figure, the effects of saturation is to limit the obtained velocity.

Let us examine the effects of deadzone nonlinearity.

Add a deadzone (-2 to +2) to system and compare the system responses with and without the deadzone nonlinearity. Note that the input is sinusoidal.
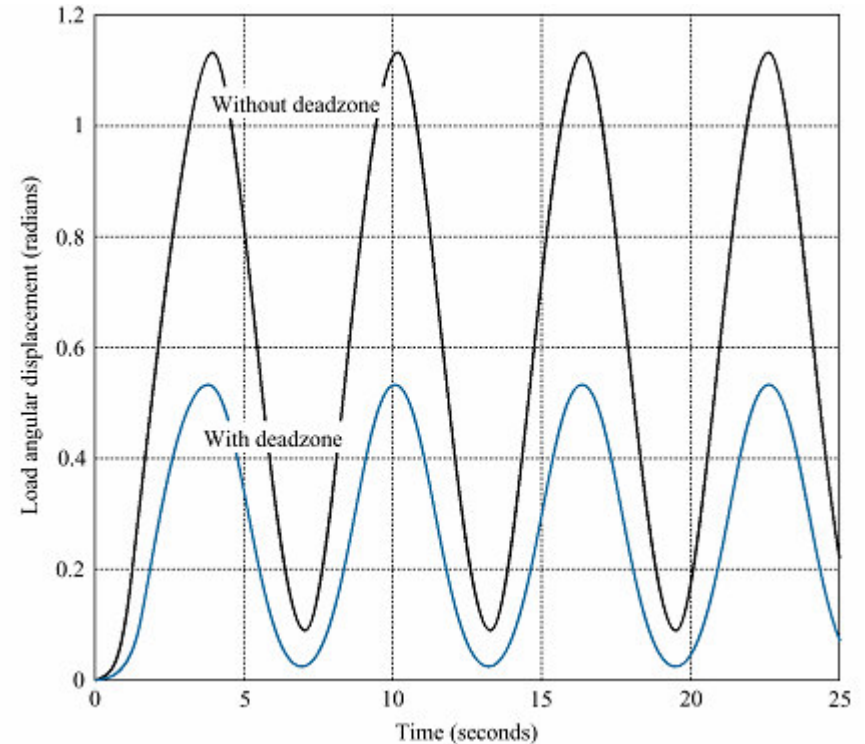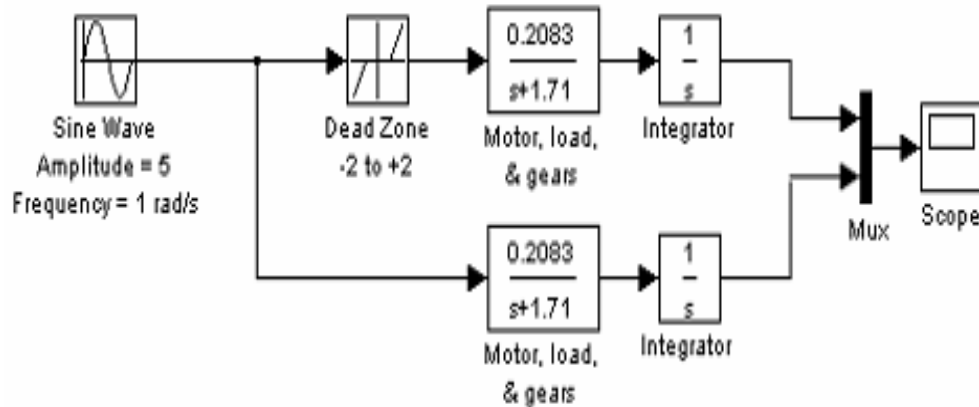
Note that, the response begins when the input voltage   to the motor exceeds a threshold.

Let us examine the effects of backlash nonlinearity.

Add a backlash to system and compare the system responses with and without the backlash nonlinearity. Note that the input is sinusoidal.
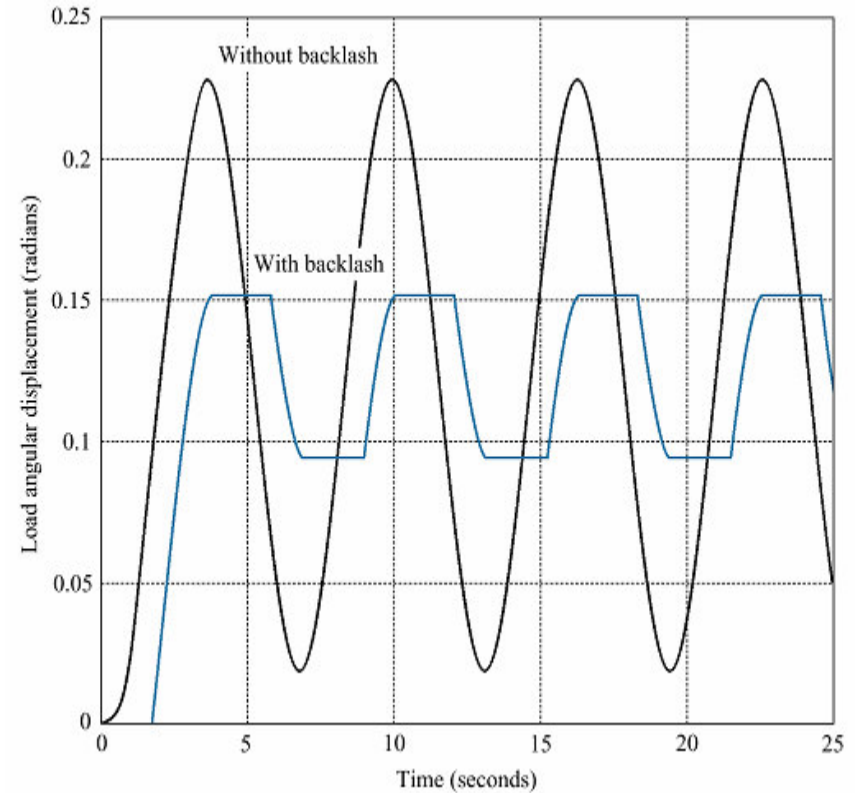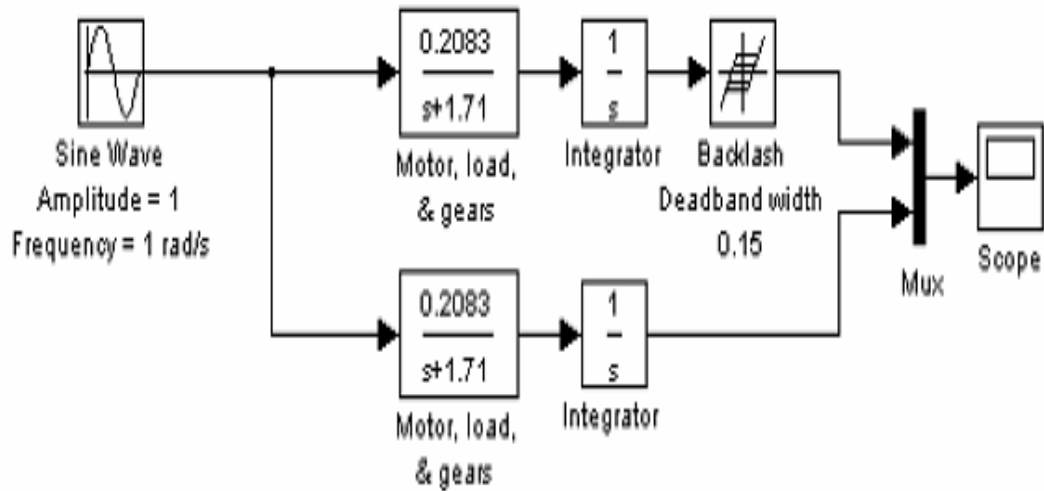
As the motor reverses direction, the output shaft remains stationary while the motor begins to reverse. The resulting response is quite different from the linear response without backlash.

LAPLACE TRANSFORM SOLUTION OF STATE EQUATIONS

In this section, we will use the Laplace transform to solve the state equations for the state and output vectors. Consider the state and output equations

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Taking the Laplace transform of both sides of the state equations yields

$$sX(s) - x(0) = AX(s) + BU(s)$$

In order to seperate X(s), replace sX(s) with sIX(s), where I is an n*n identity matrix, and n is the order of the system. Combining all of the X(s) terms, we get

$$(sI - A)X(s) = x(0) + BU(s)$$

Solving for X(s) by premultiplying both sides of the last equation by (sI-A)$^{-1}$ yields

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

$$= \frac{adj(sI - A)}{\det(sI - A)}[x(0) + BU(s)]$$

Tkanig the Laplace transform of the output equation yields

$$Y(s) = CX(s) + DU(s)$$

<u>Eigenvalues and Transfer Function Poles</u> : We saw that the poles of the transfer function determine the nature of the transient response of the system. Is there an equivalent quantity in the state-space representation that yields the same information? The roots of det(sI-A)=0 is the eigenvalues of the system matrix A. Let us show that the eigenvalues are equal to the poles of the system's transfer function. Let the output, Y(s), and the input, U(s), be scalar quantities Y(s) and U(s), respectively. Further, conform the definition of a transfer function, let x(0), the initial state vector, equal 0, the null vector. Substituting the equation

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

$$= \frac{adj(sI - A)}{\det(sI - A)} [x(0) + BU(s)]$$

into equation $Y(s) = CX(s) + DU(s)$ and solving for the transfer function Y(s)/U(s) yields

$$\frac{Y(s)}{U(s)} = C \left(\frac{adj(sI - A)}{\det(sI - A)} \right) B + D$$

$$= \frac{C adj(sI - A) B + D \det(sI - A)}{\det(sI - A)}$$

The roots of the denominator of this equation are the poles of the system. Since, the roots of X(s) and Y(s)/U(s) are identical, the system poles equal the eigenvaluses. Hence, if a system is represented in state-space, we can find the poles from det(sI-A)=0. The following example demonstrates solving the state equation using the Laplace transform as well as finding the eigenvalues and system poles.

Example : Given the system represented in state,space by equations

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \quad , \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

do the following :

a. Solve the preceding state equation and obtain the output for the given input.

b. Find the eigenvalues and the system poles.

Solution : a) Remember the equation $X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$

$$(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix} \Rightarrow (sI - A)^{-1} = \frac{\begin{bmatrix} (s^2 + 9s + 26) & s+9 & 1 \\ -24 & s^2 + 9s & s \\ -24s & -26s + 24 & s^2 \end{bmatrix}}{s^3 + 9s^2 + 26s + 24} \quad , \quad U(s) = \frac{1}{s} \quad \text{we get}$$

$$\left. \begin{aligned} X_1 &= \frac{(s^3 + 10s^2 + 37s + 29)}{(s+1)(s+2)(s+3)(s+4)} \\ X_2 &= \frac{(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)} \\ X_3 &= \frac{s(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)} \end{aligned} \right\}$$

$$Y(s) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s) \Rightarrow Y(s) = \frac{(s^3 + 12s^2 + 16s + 5)}{(s+1)(s+2)(s+3)(s+4)} = \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4}$$

where a pole at -1 canceled a zero at -1. Taking the inverse Laplace transform :

$$y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}$$

b) The roots of det(sI-A)=0 give us both the poles of the system and the eigenvalues which are -2, -3 and -4.

TIME DOMAIN SOLUTION OF STATE EQUATIONS

We now look at another technique for solving the state equations. Rather than using the Laplace transform, we solve the equations directly in the time domain using a method closely allied to classical solution of differential equaitons. We will find that the final solution consist of two parts that are different from the forced and natural responses.

The solution in the time domain is given directly by

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$= \Phi(t) x(0) + \int_0^t \Phi(t-\tau) Bu(\tau) d(\tau)$$

where $\Phi(t) = e^{At}$ by definition, and which is called the **state-transition matrix**.

Notice that the first termon the right hand side of the equation is the response due to the initial state vector $x(0)$. Notice also that it is the only term dependent on the initial state vector and not the input. We call this part of the response the ***zero-input response***, since it is the total response if the input is zero. The second term, called the *convolution integral*, is dependent only on the input, u, and the input matrix, B, not the initial state vector. We call this part of response the **zero-state response**, since it is total response if the initial state vector is zero.

Let us examine the form the elements of $\Phi(t)$ take for linear, time invariant systems. The equation $X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} B U(s)$, is the transform of $\Phi(t)x(0)$, zero input response. Thus, for the unforced system, take the Laplace transform of x(t) yields

X(s)=(sI-A)$^{-1}$x(0)

From which we can see that (sI-A)$^{-1}$ is the *Laplace transform of the state-transition matrix*. We have already seen that the denominator of (sI-A)$^{-1}$ is a polynomial in s whose roots are the system poles. This polynomial is found from the equation det(sI-A)=0. Since,

$$L^{-1}\left[(sI - A)^{-1}\right] = L^{-1}\left[\frac{adj(sI - A)}{\det(sI - A)}\right] = \Phi(t)$$

where L^{-1} represents the inverse Laplace transform operator.

Example : For the state equation and initial state vector shown in the equations, where u(t) is a unit step, find the state transition matrix and then solve for x(t).

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad , \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution : Find the eigenvalues using det(sI-A)=0. $s^2+6s+8=0$, from which $s_1=-2$ and $s_2=-4$. Since each term of the state-transition matrix is the sum of responses generated by the poles (or eigenvalues), we assume a state-transition matrix of the form

$$\Phi(t) = \begin{bmatrix} (K_1 e^{-2t} + K_2 e^{-4t}) & (K_3 e^{-2t} + K_4 e^{-4t}) \\ K_5 e^{-2t} + K_6 e^{-4t}) & (K_7 e^{-2t} + K_8 e^{-4t}) \end{bmatrix}$$

With $\Phi(t)=0$, we get 8 simultaneous equations to find the values of the constants K_i's. After the values of all constants are calculated, we get

$$\Phi(t) = \begin{bmatrix} (2e^{-2t} + e^{-4t}) & (\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t}) \\ (-4e^{-2t} + 4e^{-4t}) & (-e^{-2t} + 2e^{-4t}) \end{bmatrix} \Rightarrow \Phi(t-\tau)B = \begin{bmatrix} (\frac{1}{2}e^{-2(t-\tau)} - \frac{1}{2}e^{-4(t-\tau)}) \\ (e^{-2(t-\tau)} + 2e^{-4(t-\tau)}) \end{bmatrix}$$

Hence the first term of the state transition matrix will be $\Phi(t)x(0) = \begin{bmatrix} (2e^{-2t} - e^{-4t}) \\ (-4e^{-2t} + 4e^{-4t}) \end{bmatrix}$

and the second term will be $\int_0^t \Phi(t-\tau)Bu(t)d\tau = \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$

The final result is found as $x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau = \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t} \end{bmatrix}$