REDUCTION OF MULTIPLE SYSTEM

In this chapter, you will learn the following:

❖How to reduce a block diagram of multiple subsystems

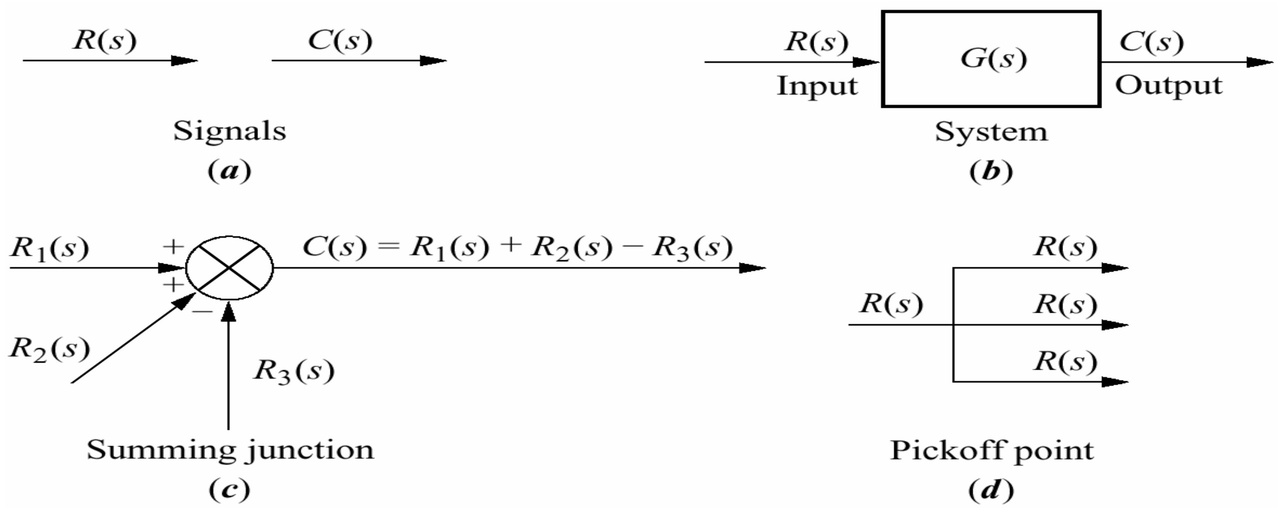
❖How to analyze and design transient response for a system consisting of multiple subsystem

❖How to represent in state-space a system consisting of multiple subsystem

Multiple subsystems are represented in two ways : as block diagrams and as signal-flow graphs. Generally, block diagrams are used for frequency domain analysis, and signal-flow graphs are used for state-space analysis. We will develop techniques to reduce each representation to a single transfer function. Block diagram algebra will be used for block diagram reduction and Mason's rule will be used to reduce signal-flow graphs.

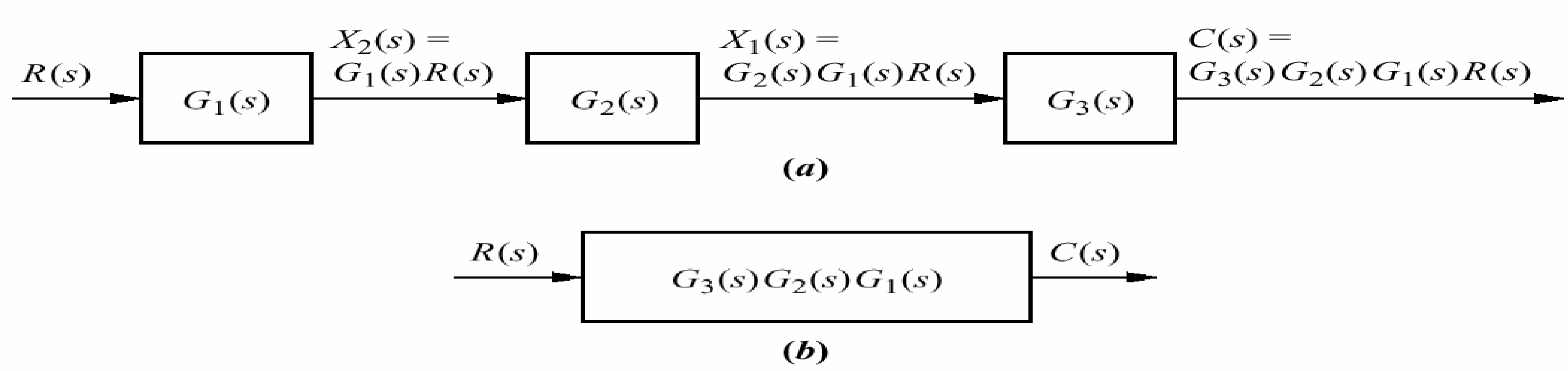
BLOCK DIAGRAMS

A subsystem is represented as a block with an input, an output and a transfer function. Many systems are composed of multiple subsystems. Whem multiple subsystems are interconnected, a few more schematic elements must be added to the block diagram. These new elements are ***summing junctions*** and ***pickoff points***. All component parts of a block diagram for a linear, time invariant system are shown in the figure. The characteristic of the summing junction shown in figure(c) is that the output signal C(s), is the algebraic sum of the input signals $R_1(s)$, $R_2(s)$ and $R_3(s)$. A pickoff point, as shown in figure(d), distributes the input signal, R(s), undiminished, to several output points.



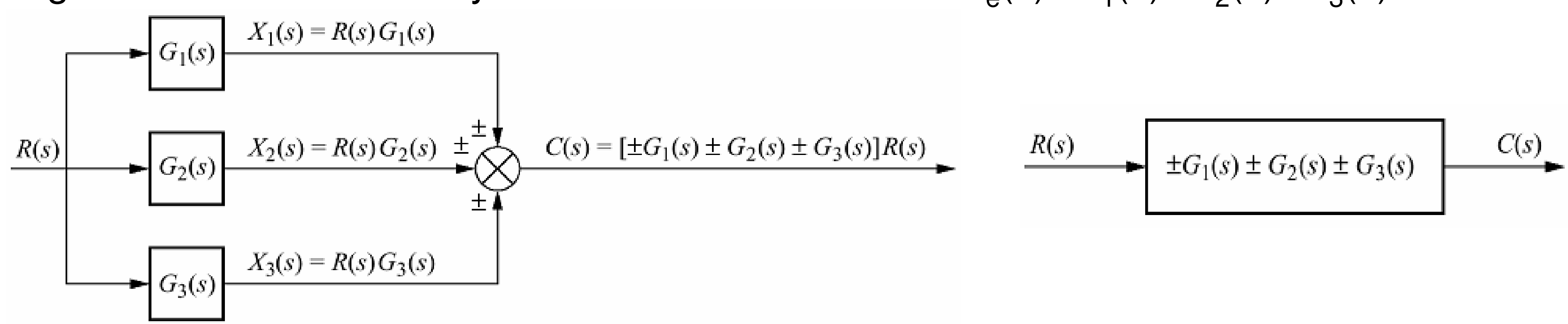
We will now examine some common topologies for interconnecting subsystems and derive the single transfer function representation for each of them.

Cascade Form : The following figure shows an example of cascaded subsystems. Intermediate signal values are shown at the output of each subsystems.



Each signal is derived from the product of the input times the transfer function. The equivalent transfer function, $G_e(s)$, shown in figure(b), is the output Laplace transform divided by the input Laplace transform from figure(a), or $G_e(s)=G_3(s)G_2(s)G_1(s)$.

Parallel Form : In the parallel form, the equivalent transfer function $G_e(s)$ is the algebraic sum of the subsystems' transfer functions. $G_e(s)=G_1(s)\pm G_2(s)\pm G_3(s)$



Feedback Form : The feedback system is the basis for our study of control system engineering. Let us derive the transfer function represents the system.

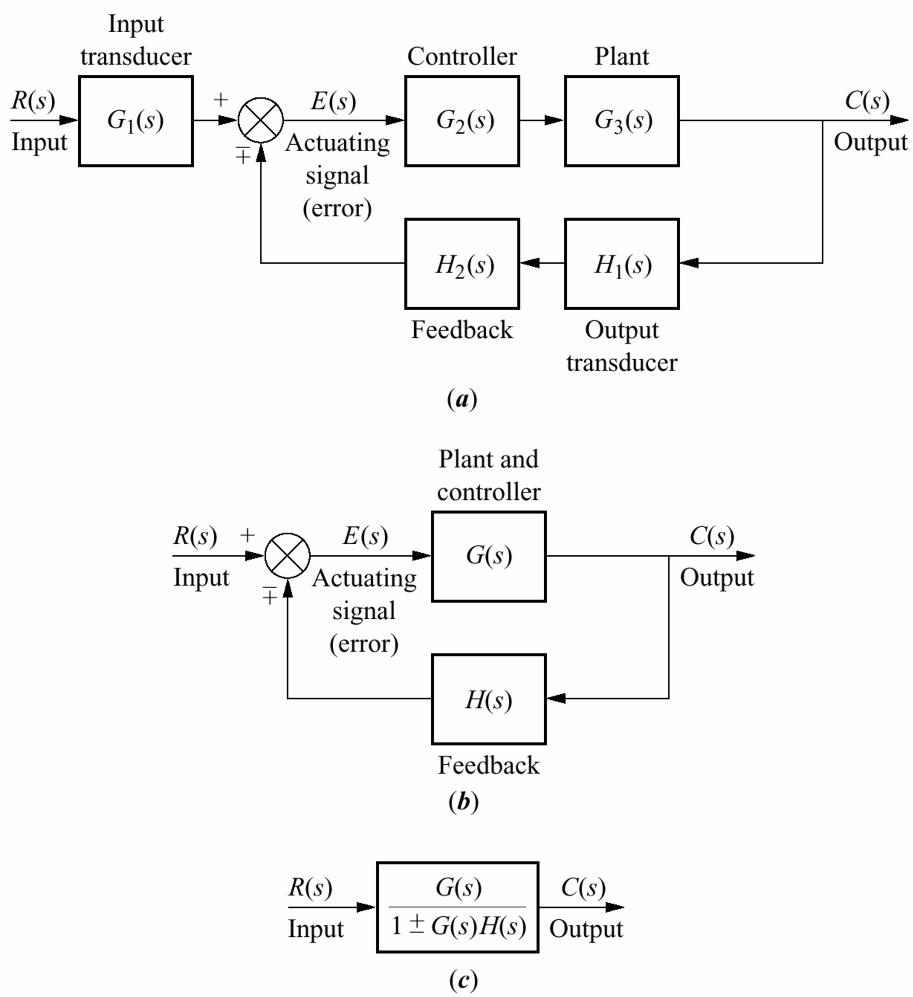


Figure 5.6

a. Feedback control system;

b. simplified model;

c. equivalent transfer function

The typical feedback system is shown in figure(a). A simplified model is shown in figure(b). Directing our attention to simplified model, **E(s) = R(s) ± C(s)H(s)**

But since C(s)=E(s)G(s), substituting E(s) in the second equation to the first equation and solving for transfer function $G_e(s)$=C(s)/R(s), we obtain the ***closed loop transfer function*** shown in figure(c),

$$G_e(s) = \frac{G(s)}{1 \pm G(s)H(s)}$$

The product G(s)H(s) is called the ***open loop transfer function*** or ***loop gain***.

So far, we have explored three different configrations for multiple subsystems. Since these three forms are combined into complex arrangements in physical systems, recognizing these topologies is a prerequisite to obtaining the equivalenttransfer function. Now, we will reduce complex systems.

Moving Blocks to Create Familiar Forms : This subsection will discuss basic block moves that can be made to order to establish familiar forms when they almost exist. In particular, it will explain how to move blocks left and right past summing junctions and pickoff points. Following figure shows equivalent block diagrams formed when transfer function are moved left or right past a pickoff point.

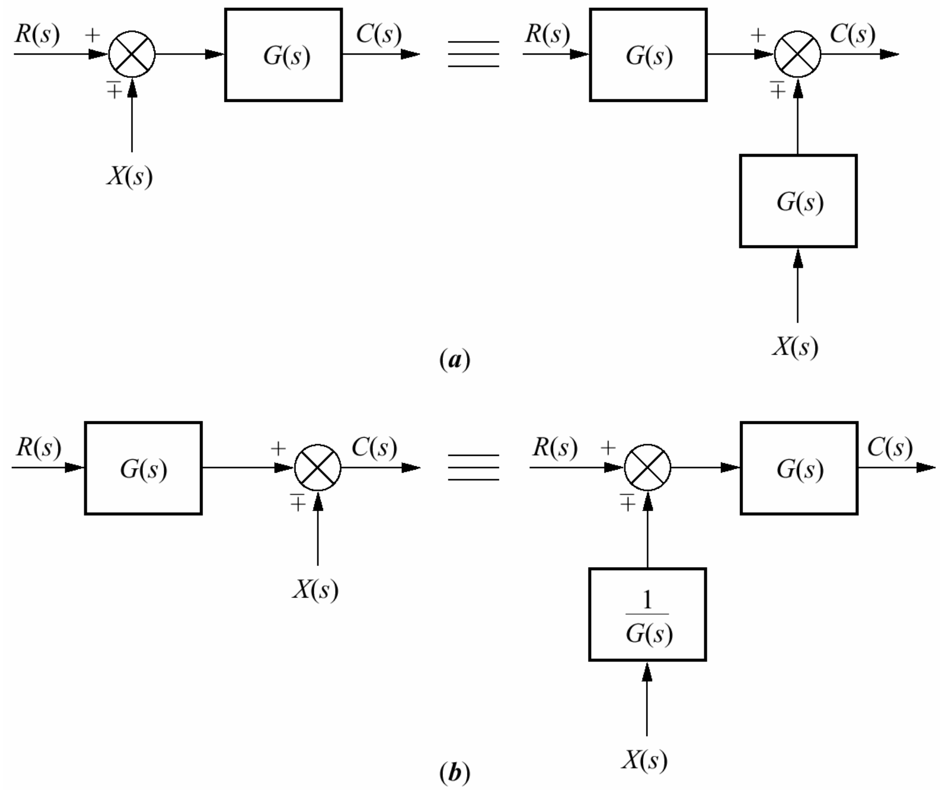


Figure 5.7
Block diagram algebra for summing junctions—equivalent forms for moving a block
a. to the left past a summing junction;
b. to the right past a summing junction

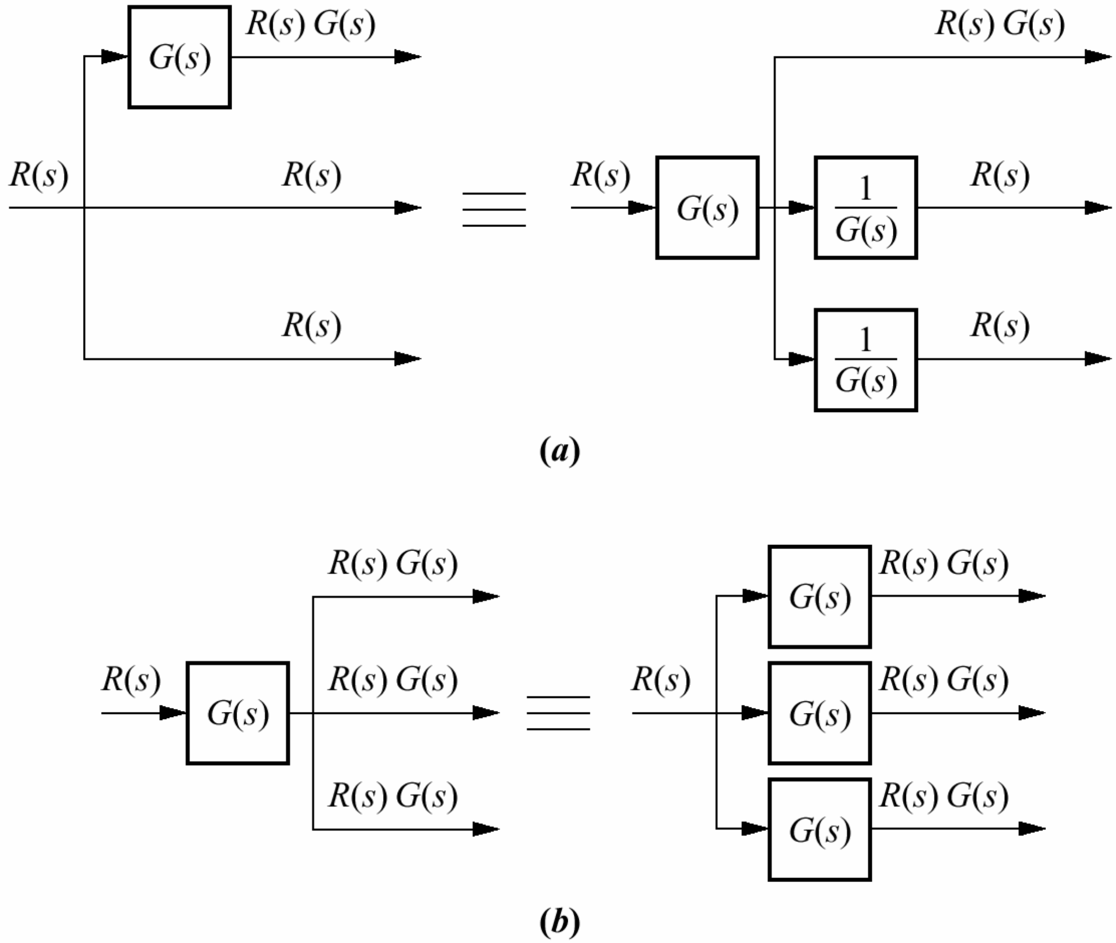
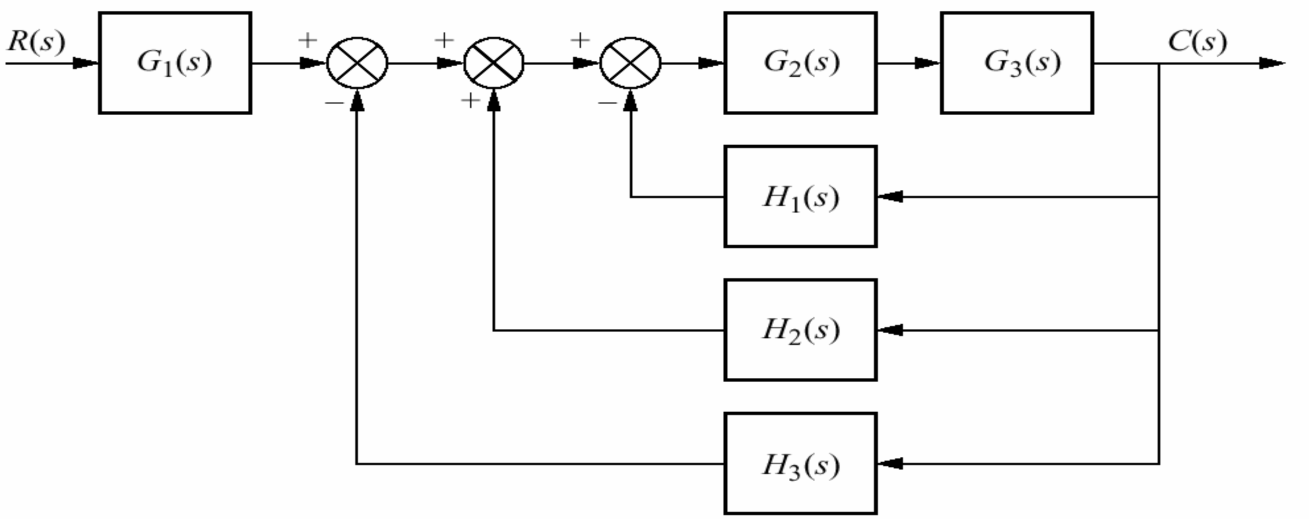
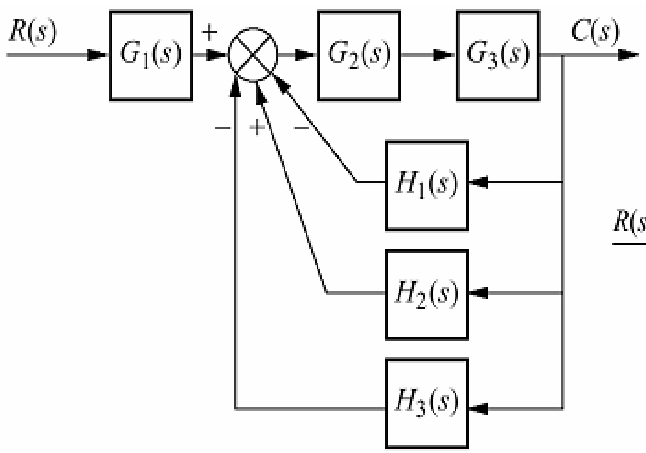


Figure 5.8
Block diagram algebra for pickoff points—equivalent forms for moving a block
a. to the left past a pickoff point;
b. to the right past a pickoff point

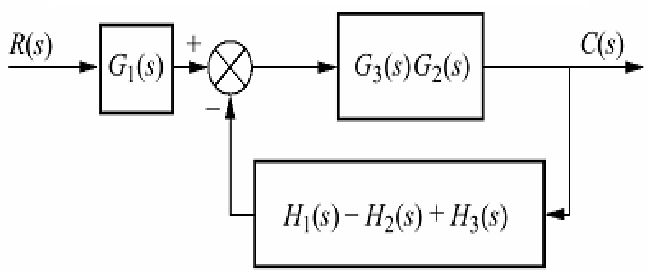
Example : Reduce the block diagram shown in figure to a single transfer function.



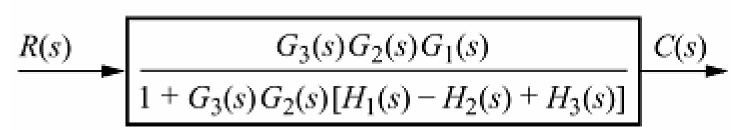
Solution : First, the three summing junction can be collapsed into a single summing junction as shown in (a). Second, recognize that three feedback functions, $H_1(s)$, $H_2(s)$, and $H_3(s)$ are connected into parallel. The equivalent function is $H_1(s)-H_2(s)+H_3(s)$. Also recognize that$G_2(s)$ and $G_3(s)$ are connected in cascade. Thus, the equivalent transfer function is the product $G_3(s)G_2(s)$ as shown in (b). Finally, the feedback system is reduced and multiplied by $G_1(s)$ to yield the equivalent transfer function shown in (c).



(a)



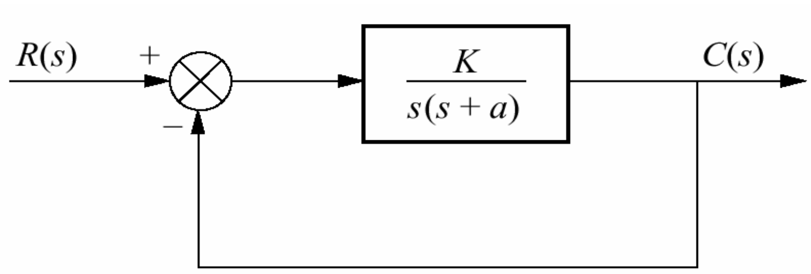
(b)



(c)

Analysis and Design of Feedback Control Systems

In this subsection we will evaluate the expressions for percent overshoot, settling time, peak time and rise time. Consider the system shown in the figure, which can model a control system such as the antenna azimuth position control system. For example, the



transfer function, K/s(s+a), can model the amplifiers, motor, load and gears. The closed loop transfer function T(s) for this system is

$$T(s) = \frac{K}{s^2 + as + K}$$

where K models the amplifier gain. As K varies, the poles move through the three ranges of operation of a second order system: overdamped, critically damped and underdamped. For example, for K between 0 and $a^2/4$, the poles of the system are real and are located at

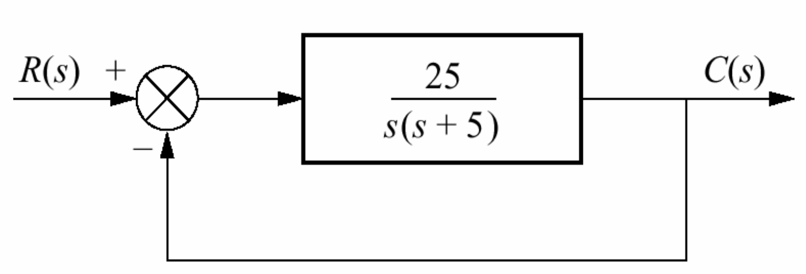
$$s_{1,2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2}$$

As K increases, the poles move along the real axes, and the system remains overdamped until $K=a^2/4$. At that gain, or amplification, both poes are real and equal, and the system is critically damped. For gains above $a^2/4$, the system is underdamped, with complex poles located at

$$s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2}$$

Now as K increases, the real part remains constant and the imaginary part increases. Thus the peak time decreasesand the percent overshoot increases, while the settling time remains constant. Let us look at two examples.

Example : For the system shown in figure, find the peak time, settling time and %PO.



Solution : The closed loop transfer function is

$$T(s) = \frac{G(s)}{1 \pm G(s)H(s)} \qquad T(s) = \frac{25}{s^2 + 5s + 25}$$

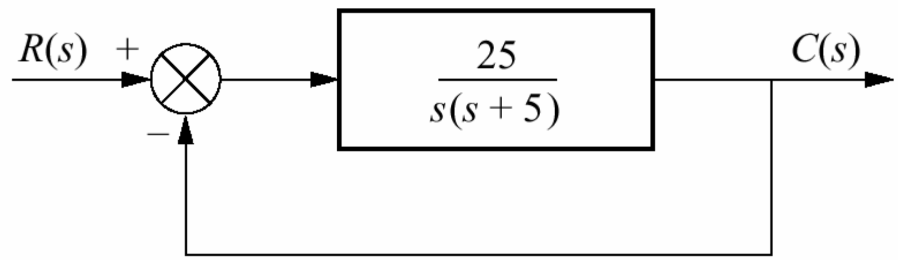
$$\omega_n = \sqrt{25} = 5 \qquad 2\zeta\omega_n = 5 \Rightarrow \zeta = 0.5$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \text{ second}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 16.303$$

$$T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ sec}$$

Example : Design the value of gain, K, for the feedback control system of figure so that the system will respond with a 10% overshoot.



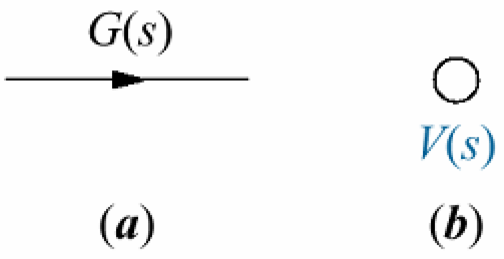
Solution : The closed loop transfer of the system is $T(s) = \dfrac{K}{s^2 + 5s + K}$

$$2\zeta\omega_n = 5 \qquad \omega_n = \sqrt{K} \qquad \zeta = \frac{5}{2\sqrt{K}}$$

Since the percent overshoot is a function only of ζ, we say that the percent overshoot is a function of K in this example. A 10% overshoot implies that ζ=0.591. Substituting this value for the damping ratio into the last equation, and solving for K yields K=17.9

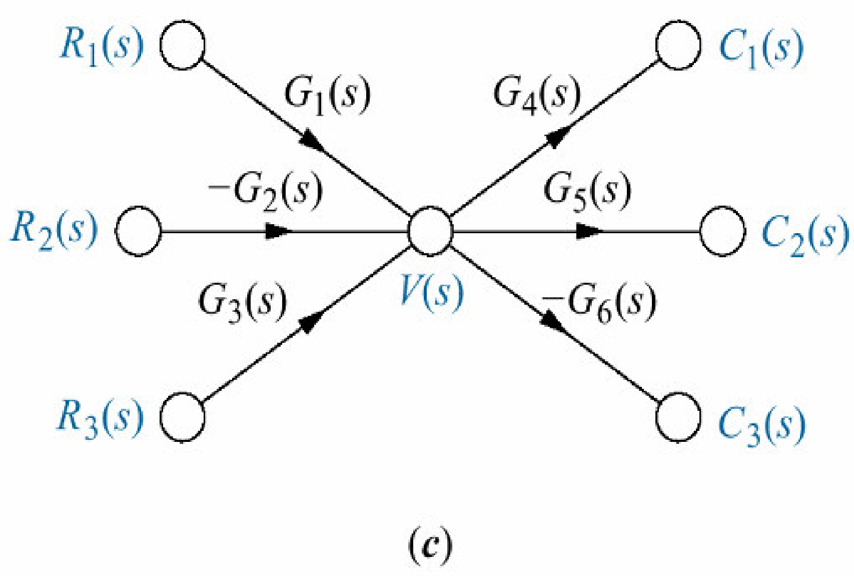
SIGNAL-FLOW GRAPHS

Signal-flow graphs are an alternative to block diagrams. Unlike block diagrams, which consist of blocks, signals, summing junctions, and pickoff points, a signal-flow graph consist only of ***branches***, which represent system, and ***nodes***, which represent signals. These elements are shown in figure (a) and (b) respectively.



A system is shown represented by a line with an arrow showing the direction of signal-flow through the system. Adjacent to the line we write the transfer function. A signal is a node with the signal's name written adjacent to the node.

Figure(c) shows the interconnection of the systems and the signals. Each signal is the sum of signals flowing into it. For example, the signal V(s),



$V(s) = R_1(s)G_1(s) - R_2(s)G_2(s) + R_3(s)G_3(s)$

For $C_2(s)$,

$$C_2(s) = V(s)G_5(s)$$

$$= R_1(s)G_1(s)G_5(s) - R_2(s)G_2(s)G_5(s)$$

$$+ R_3(s)G_3(s)G_5(s)$$

$$C_3(s) = - V(s)G_6(s)$$

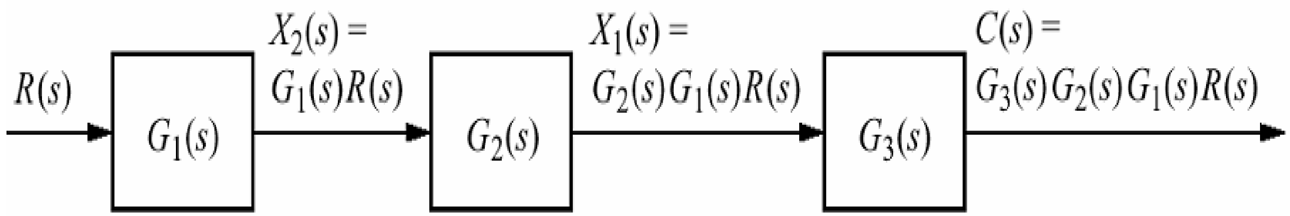
$$= - R_1(s)G_1(s)G_6(s) + R_2(s)G_2(s)G_6(s)$$

$$- R_3(s)G_3(s)G_6(s)$$

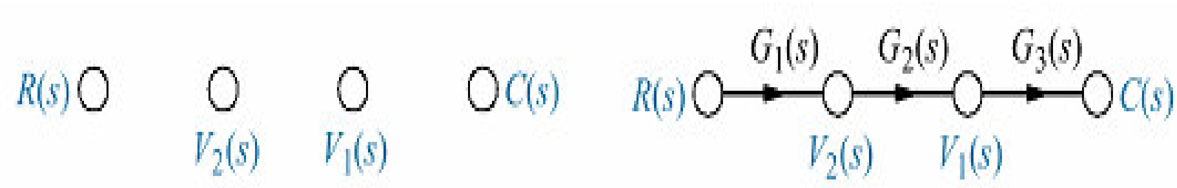
To show parallelity between block diagrams and signal-flow graphs, we will take some of the block diagram forms from the previous section and convert them to signal-flow graphs in the following example. In each case we will first convert the signals to nodesand then interconnect the nodes with system branches.

Example : Convert the cascaded, parallel and feedback forms of the block diagrams shown in figure into signal-flow graphs.

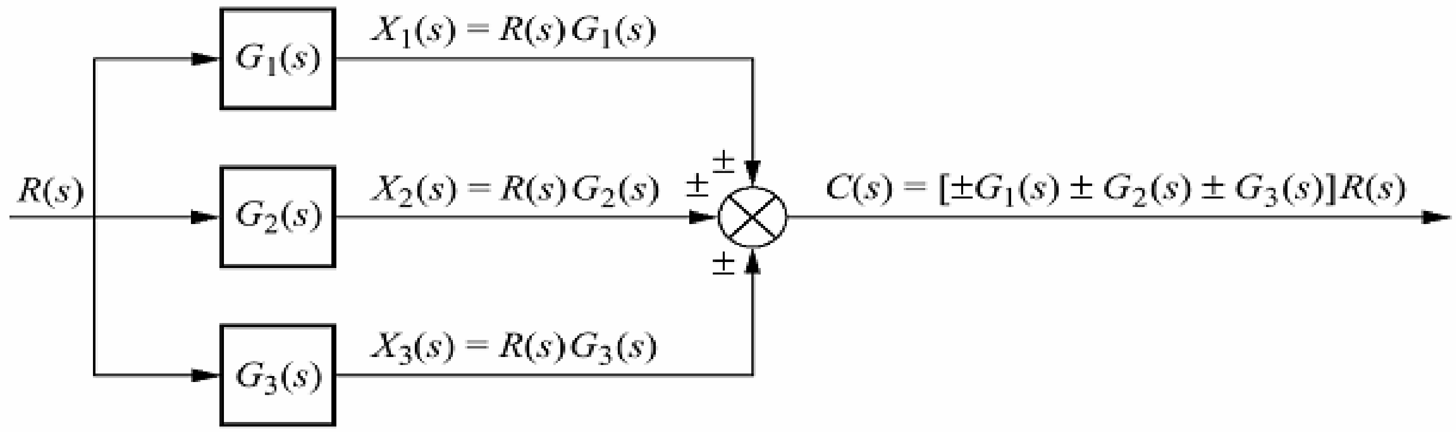
a)



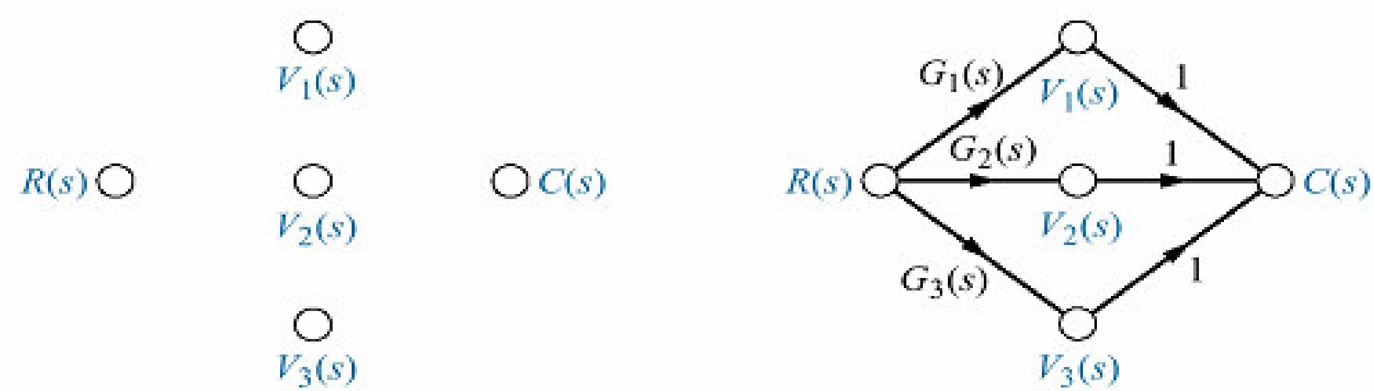
Solution : Begin by drawing the signal nodes. Next interconnect the signal nodes with system branches.



b)

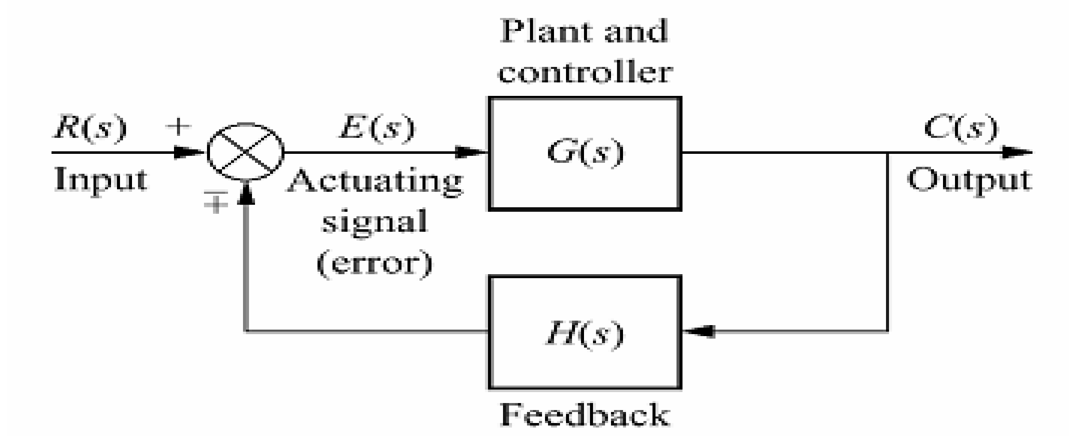


$G_1(s)$
$G_2(s)$
$G_3(s)$
$R(s)$
$X_1(s) = R(s)G_1(s)$
$X_2(s) = R(s)G_2(s)$
$X_3(s) = R(s)G_3(s)$
$\pm$
$\pm$
$\pm$
$C(s) = [\pm G_1(s) \pm G_2(s) \pm G_3(s)]R(s)$

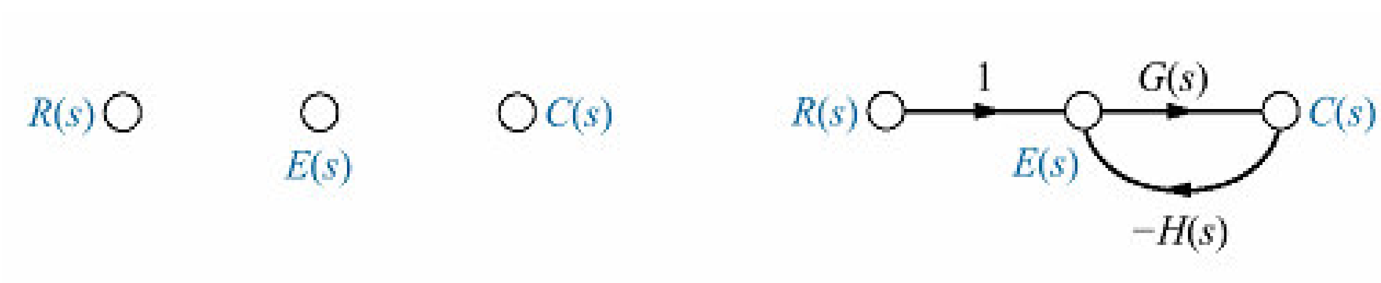


$V_1(s)$
$R(s)$
$V_2(s)$
$C(s)$
$V_3(s)$
$G_1(s)$
$V_1(s)$
1
$G_2(s)$
1
$R(s)$
$V_2(s)$
$C(s)$
$G_3(s)$
1
$V_3(s)$

c)

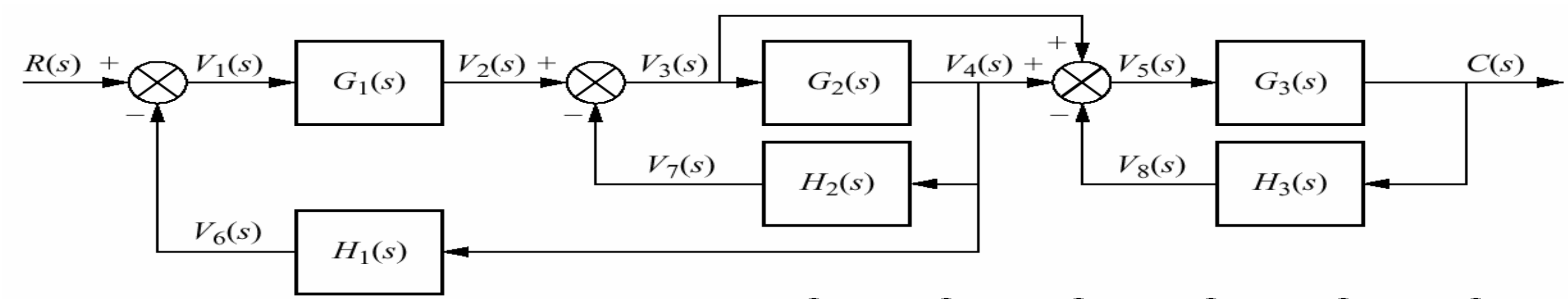


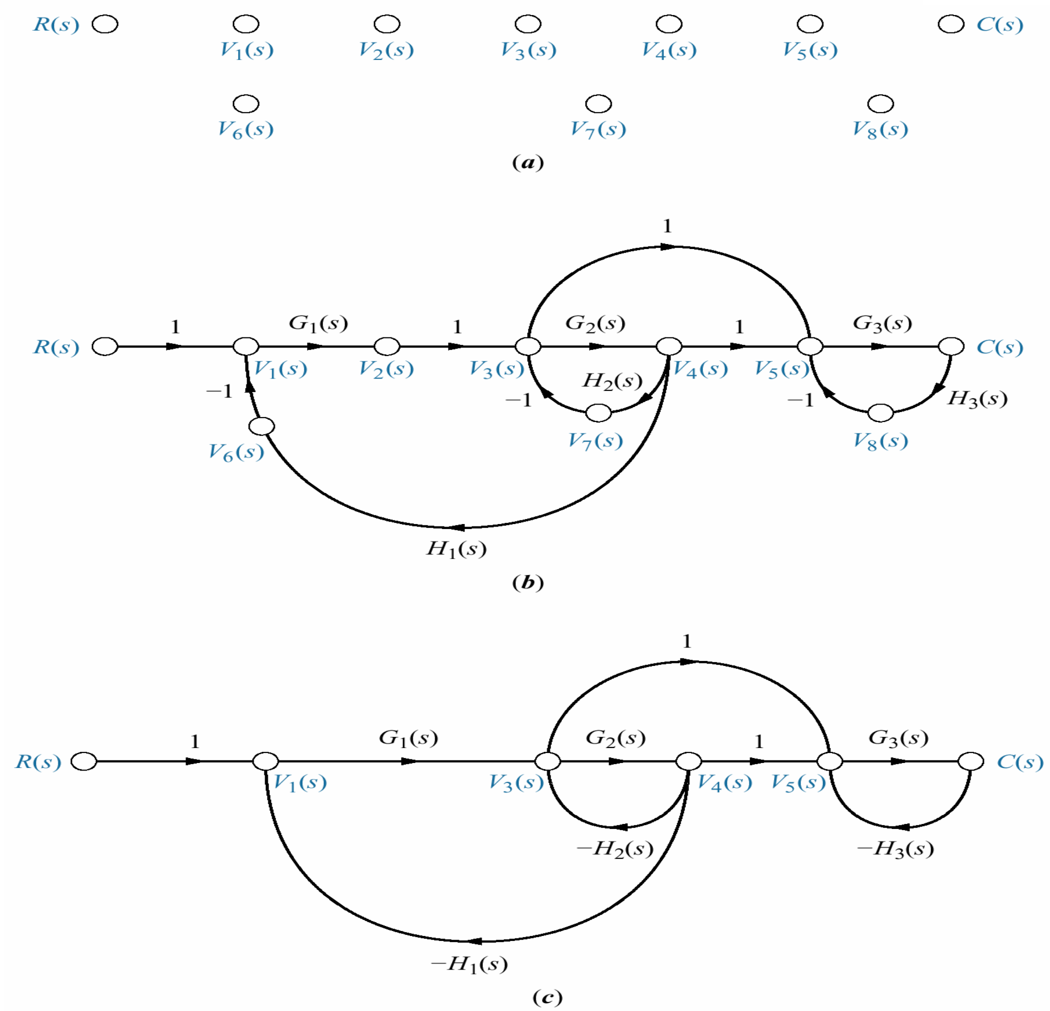
Plant and controller
R(s)
+
Input
∓
E(s)
Actuating signal (error)
G(s)
C(s)
Output
H(s)
Feedback



R(s)
E(s)
C(s)
R(s)
1
G(s)
C(s)
E(s)
−H(s)

Example : Convert the block diagram shown in the figure to a signal flow diagram





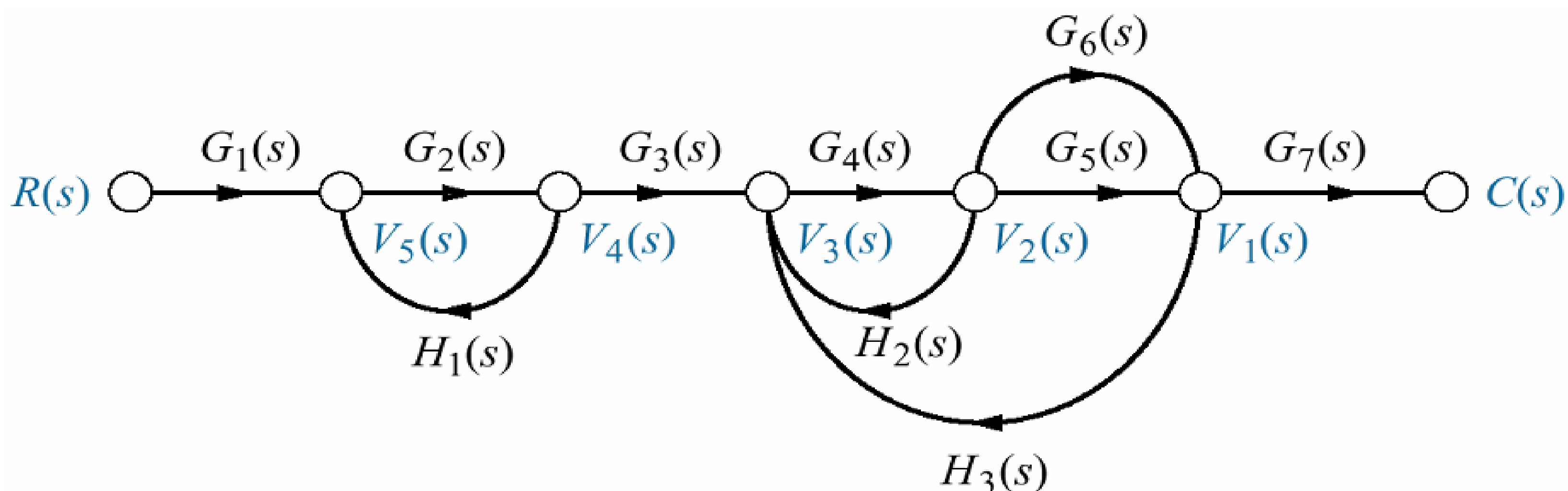
Solution : Begin by drawing the nodes as figure(a). Next interconnect the nodes as figure(b). Notice that the negative signs at the summing junction of the block diagram are represented by the negative transfer function of the signal flow diagram. Finally, if desired, simplify the signal-flow graph to the one shown in figure(c) by eliminating signals that have a single flow in and a flow out, such as $V_2(s)$, $V_7(s)$ and $V_8(s)$.

MASON'S RULE

Earlier in this lecture, we discussed how to reduce block diagrams to single transfer functions. Now we are ready to discuss a technique for reducing signal-flow graphs to single transfer functions that relate output of a system to its input.

Some Definitions Related Mason's Formulation :

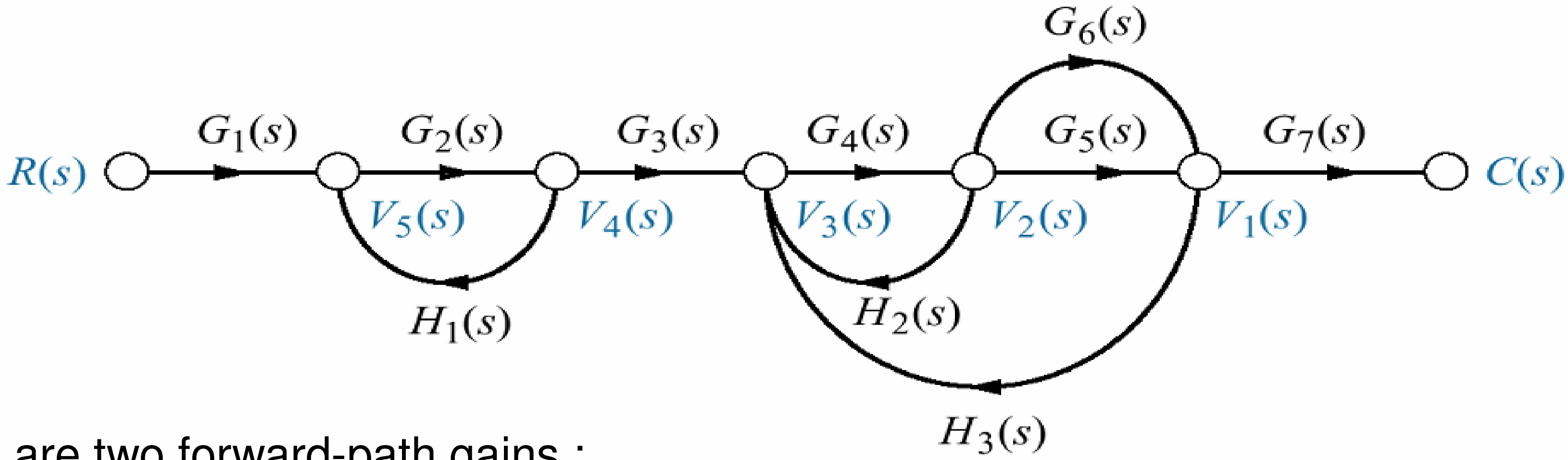
Loop Gain : The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once. For examples of loop gains, see following figure.



There are four loop gains :

1. $G_2(s)H_1(s)$ 2. $G_4(s)H_2(s)$ 3. $G_4(s)G_5(s)H_3(s)$ 4. $G_4(s)G_6(s)H_3(s)$

Forward Path Gain : The product of gains found by traversing a path from the input node to the output node of signal-flow graph in the direction of signal flow. Exapmles of forward-path gains are also shown the same figure.



There are two forward-path gains :

1.$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$ 2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

Non-touching Loop Gain : The product of loop gains from nontouching loops taken

two,three, four, or more at a time. In the above figure, the product of loop gain G2(s)H1(s)and loop gain G4(s)H2(s) is a nontouching-loop gain taken two at a time. In summary all three of nontouching loop gains taken two at a time are

1.$[G_2(s)H_1(s)][G_4(s)H_2(s)]$ 2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$

In this example there are no nontouching loop gains taken three at a time since three nontouching loop do not exist in the example.

After this definitions, we are ready to state Mason’s rule.

MASON’S RULE

The transfer function, C(s)/R(s), of a system represented by a signal-flow graph is

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

where

k = number of forward paths

T_k = the *k*th forward-path gain

Δ = 1 - **Σ**(loops gains) + **Σ**(nontouching-loop gains taken two at a time)

- **Σ**(nontouching-loop gains taken three at a time)

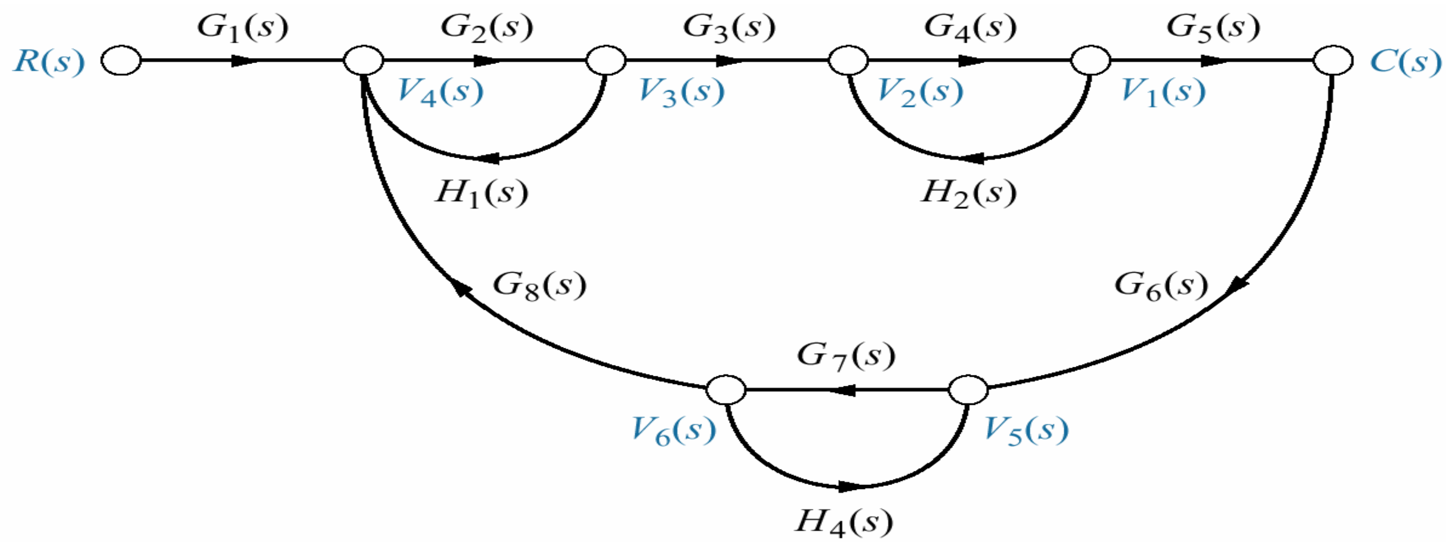
+ **Σ**(nontouching-loop gains taken four at a time)

- ……………………..

Δ_k = Δ – **Σ**(loop gain terms in Δ that touch the *k*th forward path)

In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the *k*th forward path.

Example : Find the transfer function, C(s)/R(s), for the signal flow graph in figure.



First, identify the *forward-path gains*. In this example there is only one :

$$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

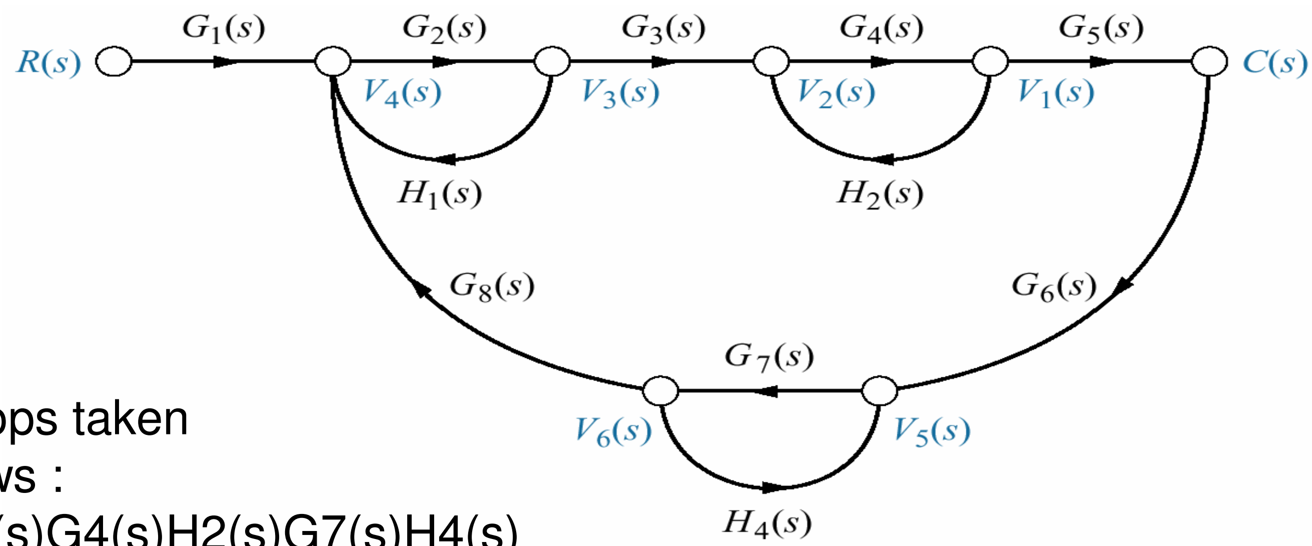
Second, identify the *loop gains*. There are four, as follows :

1.$G_2(s)H_1(s)$ 2. $G_4(s)H_2(s)$ 3.$G_7(s)H_4(s)$ 4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Third, identify the nontouching loops taken two at a time. We can see that loop 1 does not touch loop 2, loop1 does not touch loop3, ans loop 2 does not touch loop3. Note that loop 1, 2 and 3 all touch loop 4. Thus, the combinations of nontouching loops taken two at a time are as follows: Loop 1 and loop 2 : $G_2(s)H_1(s)G_4(s)H_2(s)$

Loop 1 and loop 3 : $G_2(s)H_1(s)G_7(s)H_7(s)$

Loop 2 and loop 3 : $G_4(s)H_2(s)G_7(s)H_4(s)$



Finally, the nontouching loops taken three at a time are as follows :
Loops 1,2 and 3 : G2(s)H1(s)G4(s)H2(s)G7(s)H4(s)

Now, we can form Δ and Δ_k.

$$\mathbf{\Delta}=1-[G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)]$$

$$+[G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_7(s) + G_4(s)H_2(s)G_7(s)H_4(s)]$$

$$-[G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]$$

We form Δ_k by eliminaitng from Δ the loop gains that touch the *k*th forward path :

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

The transfer function is

$$G(s)=\frac{T_1\Delta_1}{\Delta}=\frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1-G_7(s)H_4(s)]}{\Delta}$$

Since there is only one forward path, G(s) consists of only one term, rather than a sum of terms, each coming from a path.

Signal-Flow Graphs of State Equations :

In this section we draw signal-flow graphs from state-equations. At first this process will help us visualize state variables. Later we will draw signal flow graphs and then write alternate representation of a system in state space.

Consider the following state and output equations :

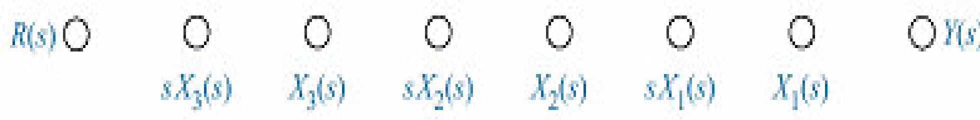
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

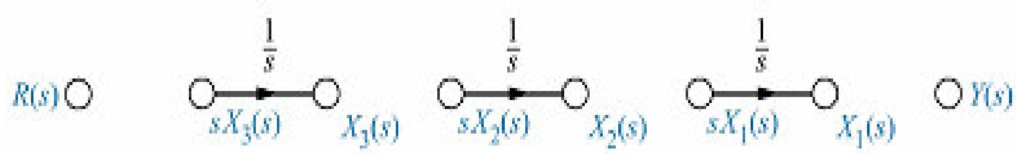
$$y = -4x_1 + 6x_2 + 9x_3$$

First, identify three nodes to be the three state variables. Also identify three nodes to be the derivative of state variables as in figure(a).



(a)

Next interconnect the state variables and their derivatives with the defining integration, 1/s, as shown in figure(b).

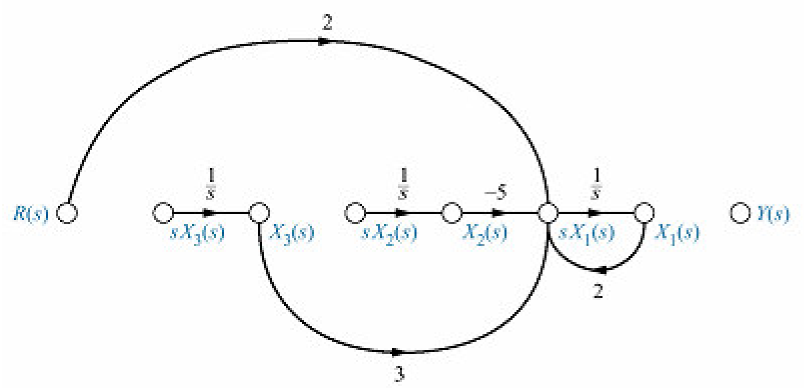


(b)

Then feed to each node the indicated signals.

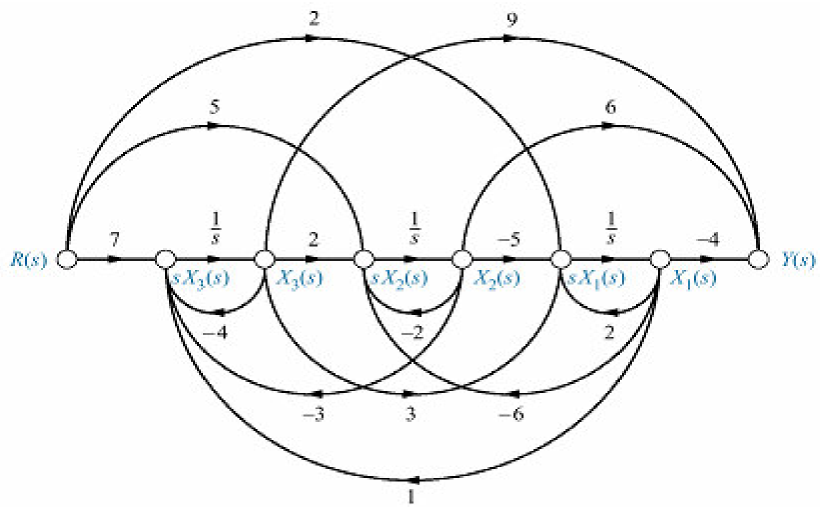
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$

For example, the derivative of x_1 receives $2x_1$, $-5x_2$, $3x_3$ and 2r. The connections for the derivative of x_1 are as follows:



(c)

If we draw the connections for the other nodes, the resulting signal flow graph will be as follows:



SIMILARITY TRANSFORMATIONS

Earlier we saw that systems can be represented with different state variables even though the transfer function relating the output to the input remains the same. The various forms of the state equations were found by manipulating the transfer function, drawing a signal flow graph, and then writing the state equation from the signal flow graph. These systems are called ***similar systems***. Although the state-space representation are different, similar systems have the same transfer function and hence the same poles and eigenvalues.

We can make transformations between similar systems from one set of state equations to another without using the transfer function and signal flow graphs. A system represented in state-space as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

can be transformed to a similar system,

$$\boxed{\begin{aligned}\dot{z} &= P^{-1}APz + P^{-1}Bu \\ y &= CPx + Du\end{aligned}}$$

where, for 2-space,

$$P = \{U_{z1} \quad \mathrm{U}_{z2}\} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \qquad x = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Pz \qquad z = P^{-1}x$$

P is a transformation matrix whose columns are the coordinates of the basis vectors of z_1z_2 space expressed as linear combinations of the x_1x_2. Let us look an example.

Example : Given the system represented in state space by the following equations

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

transform the system to a new state variables, z, where the new state variables are related to the original state variables, x, as follows

$$z_1 = 2x_1$$

$$z_2 = 3x_1 + 2x_2$$

$$z_3 = x_1 + 4x_2 + 5x_3$$

Solution : Because of $z = P^{-1}x$, note that the elements of P^{-1} are $P^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

Anymore we can calculate all other matrices :

$$P^{-1}AP = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix} , \quad P^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} , \quad CP = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}$$

Therefore, the transformed system is

$$\dot{z} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix} z$$

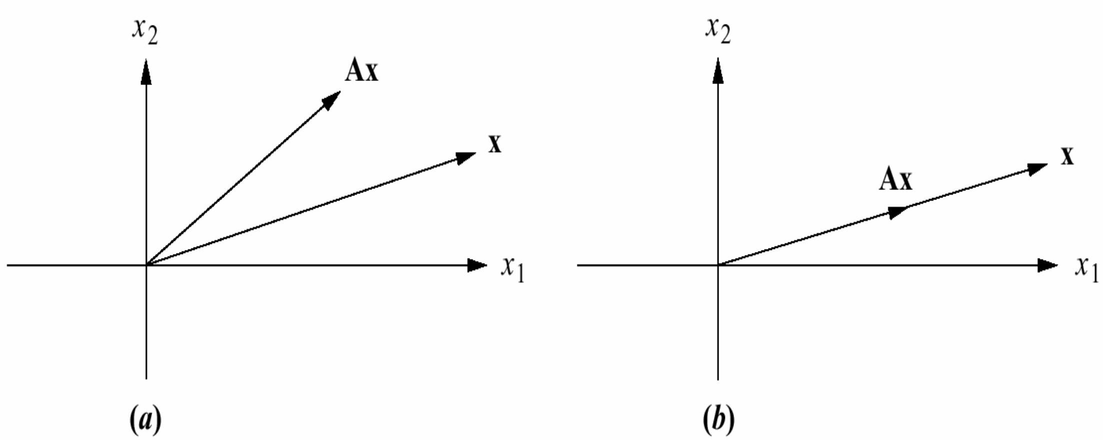
Diagonalizing a System Matrix : A diagonal system matrix has the advantage that each state equation is a function of only one state variable. Hence, each differential equation can be solved independently of other equations. We say that the equation are **decoupled**. We can decouple a system using matrix transformations. If we find the correct P matrix, the transformed system matrix $P^{-1}AP$ will be a diagonal matrix. Thus, we are looking for a transformation matrix to another state-space that yields a diagonal matrix in that space. This new state space also has basis vectors that lie along its state variables. We give a special name to any vectors that are collinear with the basis vectors of the new system that yields a diagonal system matrix: they are called ***eigenvectors***.

First, let's define eigenvectors and then we will show how to diagonalize a matrix.

Eigenvector : The eigenvectors of the matrix A are all vectors, $x_i \neq 0$, which under the transformation A become multiples of themselves; that is,

$$\mathbf{Ax_i = \lambda_i x_i}$$

where λ_i's are constants. The figure below shows this definition of eigenvectors. If A is not collinear with x after transformation, as in figure(a), x is not an eigenvector. If Ax is collinear with x after the transformation, as in figure(b), x is an eigenvector.



Eigenvalue : The eigenvalues of the matrix A are the values of λ_i that satisfy the following equation for x_i=0,

$$\mathbf{Ax_i = \lambda_i x_i}$$

To find the eigenvectors, we rearrange tihs equation. Eigenvectors, x_i, satisfy

$$0=(\lambda_i I-A)x_i$$

Solving for x_i by premultiplying both sides by $(\lambda_i I-A)^{-1}$ yields $x_i = (\lambda_i I - A)^{-1}0 = \dfrac{adj(\lambda_i I - A)}{\det(\lambda_i I - A)}0$

Since $x_i \neq 0$, a nonzero solution exists if $\det(\lambda_i I-A)=0$ from which λ_i can be found.

Now we are ready to show how to find the eigenvectors x_i. First we find eigenvalues, λ_i, using $\det(\lambda_i I-A)=0$, and then we use the equation $\mathbf{Ax_i = \lambda_i x_i}$ to find the eigenvectors.

Example : Find the eigenvectors of the matrix $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$

Solution : First use $\det(\lambda_i I-A)=0$ to find the eigenvalues, λ_i : $\det(\lambda I - A) = \left\| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right\|$

$\det(\lambda I - A) = \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} = \lambda^2 + 6\lambda + 8$ from which eigenvalues are λ = -2 and -4.

For eigenvectors, we must use the equation $\mathbf{Ax_i = \lambda_i x_i}$ for -2 and -4. For -2, we have:

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{or} \quad \begin{aligned} -3x_1 + x_2 &= -2x_1 \\ x_1 - 3x_2 &= -2x_2 \end{aligned}$$

from which $x_1=x_2$. Thus, $x = \begin{bmatrix} c \\ c \end{bmatrix}$. Using the other eigenvalue, -4, we have $x = \begin{bmatrix} c \\ -c \end{bmatrix}$

One choice of the eigenvectors is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We now show that if the eigenvectors of the matrix A are chosen as the basis vectors of a transformation, P, the resulting system matrix will be diagonal. Let the transformation matrix P consist of the eigenvectors of A, x_i. $P=[x_1,x_2,x_3,\ldots\ldots.,x_n]$

Since x_i are eigenvectors, $Ax_i=\lambda x_i$, which can be written equivalently as a set of equations expressed by

$$AP = PD$$

Where D is a diagonal matrix consisting of λ_i's are eigenvalues, along the diagonal, and $P=[x_1,x_2,x_3,\ldots\ldots.,x_n]$. Solving the equation AP = PD for D by premultiplying by P^{-1}, we get

$$\boxed{D = P^{-1}AP}$$

which is the system matrix of transformed similar system.

In summary, under the transformation P, consisting of eigenvectors of the system matrix, the transformed system is diagonal, with the eigenvalues of the system along the diagonal. The transformed system is identical to that obtained using partial-fraction expansion of the transfer function with dinstict real roots.

In the following example, we we will find eigenvectors of a second order system. Let us continue with this problem amd diagonalize system matrix.

Example : Given the the system by the following equations, find the diagonal system that is similar.

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} x$$

Solution : First find eigenvalues and eigenvectors. This step is performed in the previous example (Note that the matrix A is same). Next form the transformation matrix P, whose columns consist of eigenvectors.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Finally, form the similar system's matrix, input matrix and output matrix respectively,

$$P^{-1}AP = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \qquad P^{-1}B = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \qquad CP = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

Transformed system whose diagonal system matrix is

$$\dot{z} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} z + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 5 & -1 \end{bmatrix} z$$

Note that the diagonal elements of the system matrix are eigenvalues of transformation matrix, P.