

# STABILITY

In this lecture you will learn the following :

- ❖ How to determine the stability of a system represented as a transfer function
- ❖ How to determine the stability of a system represented in state-space
- ❖ How to determine system parameters to yield stability

**Stability** is the most important system specification. If a system is unstable, transient response and steady-state error are moot points.

**What is the stability?** There are many definitions for stability, depending upon the kind of system or point of view. In this section we limit ourselves to linear time invariant (LTI) systems. The total response of a system is the sum of the forced and natural response,

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

Using these concepts, we present the following definitions of **stability**, **instability** and **marginal stability** :

A linear time invariant system is **stable** if the natural response approaches to zero as time approaches to infinity.

A linear time invariant system is **unstable** if the natural response grows without bound as time approaches to infinity.

A linear time invariant system is **marginally stable** if the natural response neither decays nor grows but remains constant or oscillates as time approaches to infinity.

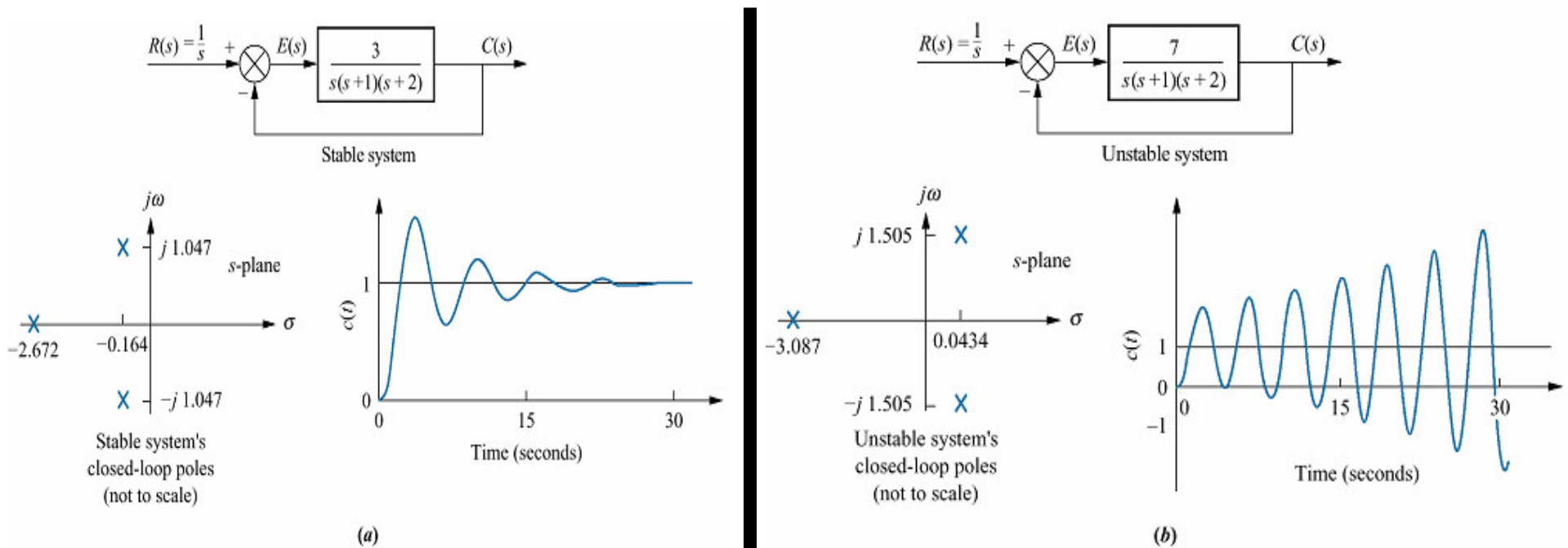
An alternative and best-known definition of stability is as follows :

**A system is stable if every bounded input yields a bounded output.**

We call this statement the bounded-input, bounded-output (BIBO) definition of stability.

**How do we determine if a system is stable?** If the closed loop system poles are in the left-half of the s-plane and hence have a negative real part, the system is stable.

Unstable systems have closed loop system transfer functions with at least one pole at the right half plane and/or poles of multiplicity greater than one on the imaginary axis. Similarly, marginally stable systems have closed loop transfer function with only imaginary axis poles of multiplicity 1 and poles in the left half plane. As an example, the unit step responses of a stable and an unstable system are shown in figure (a) and (b).



**Figure 6.1**  
 Closed-loop poles and response:  
 a. stable system;  
 b. unstable system

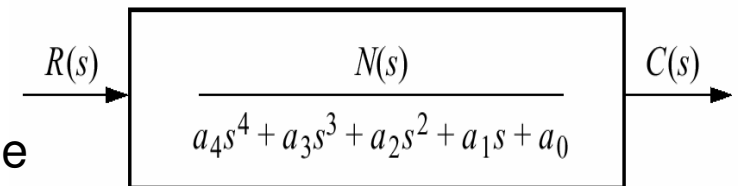
# Routh-Hurwitz Criterion

In this section we will learn a method that yields stability information without the need to solve for closed loop system poles. The method is called the *Routh-Hurwitz criterion for stability* (Routh, 1905).

The method requires two steps : (1) Generate a data table called a Routh table and (2) interpret the *Routh table* to tell how many closed loop system poles are in the left half plane, the right half plane, and on the imaginary axis.

## Generating a Basic Routh Table :

Look at the equivalent closed-loop transfer function shown in figure. Since we are interested in the system poles, we focus our attention on the denominator. We first create the Routh table shown in Table.



Begin by labelling the rows with powers of  $s$  from the highest power of the denominator of the closed loop transfer function to  $s^0$ . Next start with the coefficient of the highest power of  $s$  in the denominator and list, horizontally in the first row, every other coefficient. In the second row list horizontally, starting with the highest power of  $s$ , every coefficient that was skipped in the first row.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$			
$s^1$			
$s^0$			

Table 6.1 Initial layout for Routh table

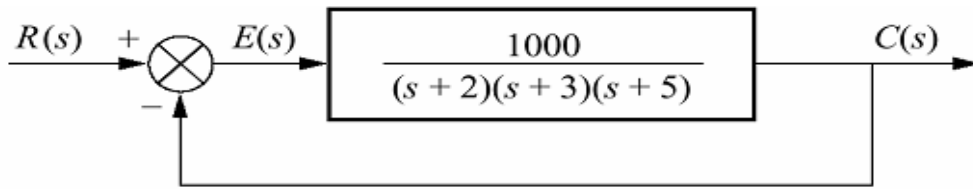
$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$			
$s^1$			
$s^0$			

The remaining entries are filled in as follows. Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row. The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right. The table is complete when all of the rows are completed down to  $s^0$ . Table 6.2 is the completed Routh table. Let us look an example.

Table 6.2 Completed Routh table

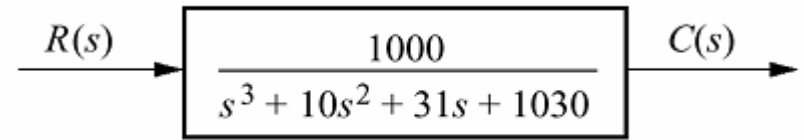
$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
$s^0$	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

**Example :** Make the Routh table for the system shown in the figure.



(a)

a. Feedback system



(b)

b. equivalent closed-loop system

**Solution :** the first step is to find the equivalent closed loop system because we want to test the denominator of this function, not the given forward transfer function, for pole location. Using the feedback formula, we obtain the equivalent system of figure(b). The Routh-Hurwitz criterion will be applied to this denominator. First label the rows with

powers of s from  $s^3$  down to  $s^0$  in a vertical column. Next form the first row of the table, using the coefficients of the denominator. Start with the coefficient of the highest power and skip every other power of s. Now form the second row with the coefficients of the denominator skipped in the previous step. Subsequent rows are formed with determinants as shown in table. For convenience any row of the Routh table can be multiplied by a positive constant without changing the values of rows below. For example, in the second row of table, the row was multiplied by 1/10 .

$s^3$	1	31	0
$s^2$	<del>10</del> 1	<del>1030</del> 103	0
$s^1$	$\frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
$s^0$	$\frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

**Interpreting the Basic Routh Table** : Now that we know how to generate the Routh table, let us see how to interpret it. Simply stated, the Routh-Hurwitz criterion declares that ***the number of roots of the polynomial that are in the right half plane is equal to the number of sign changes in the first column.*** If the closed loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table. For example, the following table has two sign changes in the first column. The first sign changes occurs from 1 in the  $s^2$  row to -72 in the  $s^1$  row. The second occurs from -72 in the  $s^1$  row to 103 in the  $s^0$  row. Thus, the system is unstable since two poles exist in the right half plane.

$s^3$	1	31	0
$s^2$	<del>10</del> 1	<del>1030</del> 103	0
$s^1$	$\frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
$s^0$	$\frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

## Routh-Hurwitz Criterion : Special Cases

Two special cases can occur : (1) The routh table sometimes will have a zero *only in the first column of a row*, or (2) the routh table sometimes will have an *entire row* that consists of zeros. Let us examine the first case.

**Zero Only in the First Column :** If the first element of a row is zero, division by zero would be required to form the next row. To avoid this phenomenon, an epsilon,  $\epsilon$ , is assigned to replace the zero in the first column. The value  $\epsilon$  is then allowed to approach zero from either the positive or negative side, after which the sign of the entries in the first column can be determined. Let us look at an example.

**Example:** Determine the stability of the closed loop transfer function  $T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$

**Solution :** The Solution is shown in the table. Begin by assembling the Routh table down

$s^5$	1	3	5
$s^4$	2	6	3
$s^3$	$\epsilon$	$\frac{7}{2}$	0
$s^2$	$\frac{6\epsilon - 7}{\epsilon}$	3	0
$s^1$	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
$s^0$	3	0	0

to the row where a zero appears only in the first column (the  $s^3$  row). Next replace the zero by a small number,  $\epsilon$ , and complete the table.



To begin the interpretation, we must first assume a sign, positive or negative, for the quantity  $\epsilon$ , table(b) shows the first column of table(a) along with the resulting signs for choices of  $\epsilon$  positive and  $\epsilon$  negative. If  $\epsilon$  is chosen negative table(b) will show a sign change from the  $s^3$  row to the  $s^2$  row, and there will be another sign change from the  $s^2$  row to the  $s^1$  row. Hence the system is unstable and has two poles in the right half plane. Alternatively, we could choose  $\epsilon$  positive. Table(b) would then show a sign change from the  $s^4$  row to the  $s^3$  row. Another sign change would occur from the  $s^3$  row to the  $s^2$  row. One result would be exactly the same as that for a positive choice for  $\epsilon$ . Thus the system is unstable, with two poles in the right half plane.

$s^5$	1	3	5
$s^4$	2	6	3
$s^3$	$\cancel{0} \epsilon$	$\frac{7}{2}$	0
$s^2$	$\frac{6\epsilon - 7}{\epsilon}$	3	0
$s^1$	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
$s^0$	3	0	0

**Table(a)**  
Completed Routh table for Example

Label	First Column	$\epsilon = +$	$\epsilon = -$
$s^5$	1	+	+
$s^4$	2	+	+
$s^3$	$\cancel{0} \epsilon$	+	-
$s^2$	$\frac{6\epsilon - 7}{\epsilon}$	-	+
$s^1$	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
$s^0$	3	+	+

**Table(b)** Determining signs in first column of a Routh table with zero as first element in a row

Entire Row is Zero : We now look at the second special case. Sometimes while making a Routh table, we find that an entire row consists of zeros because there is an even polynomial that is a factor of the original polynomial. This case must be handled differently from the case of a zero in only the first column of a row. Let us look at an example that demonstrates how to construct and interpret the Routh table when an entire row of zeros is present.

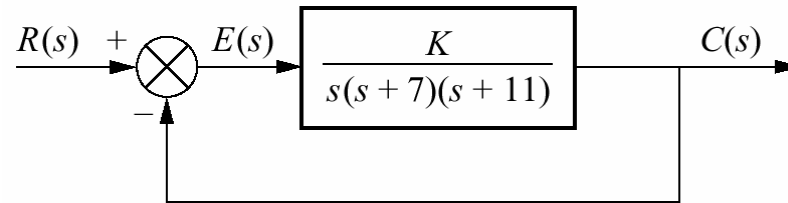
**Example :** Determine the number of right-half-plane poles in the closed transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

**Solution :** Start by forming the Routh table for the denominator. At the second row, we multiply through by 1/7 for convenience. We stop at the third row, since the entire row consists of the zeros, and use the following procedure. First we return to row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients. The polynomial will start with the power of s in the label column and continue by skipping every other power of s. Thus the polynomial formed for this example is  $P(s) = s^4 + 6s^2 + 8$ . Next we differentiate the polynomial with respect to s, and obtain  $\frac{dP(s)}{ds} = 4s^3 + 12s + 0$ . Finally we use the coefficients of the last equation to replace the row of zeros. Again, for convenience, the third row is multiplied by 1/4 after replacing zeros. There is no any right-half-plane pole. **WHY?**

$s^5$	1	6	8
$s^4$	<del>1</del>	<del>6</del>	<del>8</del>
$s^3$	<del>1</del>	<del>3</del>	<del>0</del>
$s^2$	3	8	0
$s^1$	$\frac{1}{3}$	0	0
$s^0$	8	0	0

Example : (Stability Design via Routh-Hurtwitz) : Find the range of the gain K, for the system shown below that will cause the system to be stable, unstable and marginally stable. Assume  $K > 0$  .



Solution : First find the closed-loop transfer function as  $T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$

Next form the Routh table :

$s^3$	1	77
$s^2$	18	$K$
$s^1$	$\frac{1386 - K}{18}$	
$s^0$	$K$	

Since K assumed positive, we see that all elements in the first column are always positive except the  $s^1$  row. This entry can be positive, zero, or negative, depending upon the value of K. If  $K < 1386$ , all terms in the first column will be positive, and since there are no sign changes, the system will have three poles at the left half plane and be *stable*. If  $K > 1386$ , the  $s^1$  term in the first column is negative. There are two sign changes,

indicating that the system has two right half plane poles and one left half plane pole, which makes the system *unstable*. If  $K = 1386$ , we have an entire row of zeros, which could signify  $j\omega$  poles. Returning the  $s^2$  row and replacing K with 1386, we form the even polynomial  $P(s) = 18s^2 + 1386$ . Differentiating with respect to s, we obtain  $dP(s)/ds = 36s$  . Replacing the row of zeros with the coefficients of the last equation, we obtain the following Routh table.

$s^3$	1	77
$s^2$	18	1386
$s^1$	<del>36</del>	
$s^0$	1386	

Since there are no sign changes from the even polynomial ( $s^2$  row) down to the bottom of the table, the even polynomial has its two roots on the  $j\omega$ -axis of the unit multiplicity. Since there are no sign changes above the even polynomial, the remaining root is in the left half plane. Therefore, the system is *marginally stable*.

# STABILITY IN STATE SPACE

Up to this point we have examined stability from the s-plane viewpoint. Now we look at stability from the perspective of state space. We know that the values of the system poles are equal to the eigenvalues of the system matrix. We stated that the eigenvalues of matrix  $\mathbf{A}$  were solutions of the equation  $\det(s\mathbf{I}-\mathbf{A})=0$ , which also yielded the poles of the transfer function. Let us now formally show that the eigenvalues and system poles have the same values.

The eigenvalues of a matrix,  $\mathbf{A}$ , are values of  $\lambda$  that permit a nontrivial solution (other than 0) for  $x$  in the equation  $\mathbf{A}x = \lambda x$ . In order to solve for the values of  $\lambda$ , that do indeed permit a solution for  $x$ , we rearrange the equation  $\mathbf{A}x = \lambda x$  as follows :  $\lambda x - \mathbf{A}x = 0$ , or

$$(\lambda\mathbf{I}-\mathbf{A})x = 0$$

Solving for  $x$  yields  $x = (\lambda\mathbf{I}-\mathbf{A})^{-1}0$  or  $x = \frac{\text{adj}(\lambda\mathbf{I}-\mathbf{A})}{\det(\lambda\mathbf{I}-\mathbf{A})}0$ . The values of  $\lambda$  are calculated by forcing the denominator to zero :

$$\det(\lambda\mathbf{I}-\mathbf{A}) = 0$$

This equation determines the values of  $\lambda$ . As you remember, we defined  $x$  as eigenvectors and the values of  $\lambda$  as the eigenvalues of the matrix  $\mathbf{A}$ . Let us now relate the eigenvalues of the system matrix,  $\mathbf{A}$ , to the system poles. In Lecture 3, we derived the equation of the system transfer function from the state equations. The system transfer function has  $\det(s\mathbf{I}-\mathbf{A})$  in the denominator because of the presence of  $(s\mathbf{I}-\mathbf{A})^{-1}$ . Thus the characteristic equation for the system is  $\det(\lambda\mathbf{I}-\mathbf{A}) = 0$ . Since the last two equations are identical, we conclude that the eigenvalues are equal to system poles.

**Example :** For the system given below, find out how many poles are in the left half plane, in the right half plane, and on the  $j\omega$ -axis.

$$\dot{x} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0]x$$

**Solution :** First form  $(sI-A)$  :

$$(sI - A) = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}$$

Now find the  $\det(sI-A)$ .  $\det(sI-A) = s^3 - 6s^2 - 7s - 52$

Using this polynomial, form the Routh table.

$s^3$	1	-7
$s^2$	<del>1</del> -3	<del>5</del> -26
$s^1$	<del><math>-\frac{47}{3}</math></del> -1	<del>1</del> 0
$s^0$	-26	

Since there is one sign change in the first column, the system has one right half plane pole and two left half plane poles. It is therefore unstable.