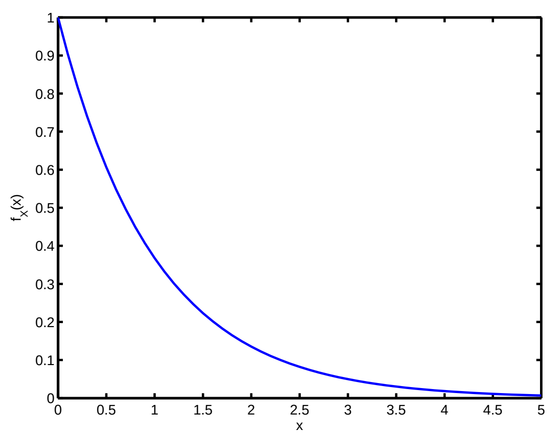
EEL 5544 Noise in Linear Systems Lecture 17

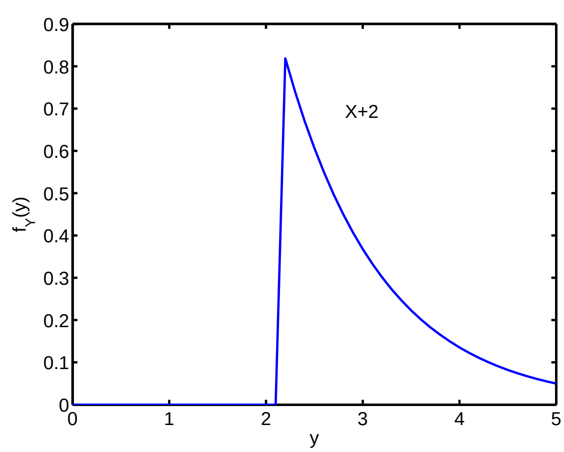
FUNCTIONS OF ONE RV

- Let's consider the basic ways in which a function can transform a random variable:
- In what follows, we will only consider the case where the input random variable is continuous
- We use the exponential random variable to illustrate the effects of different transformations
- Let X be an exponential random variable with parameter 1 and $f_X(x) = e^{-x}u(x)$



1. Functions of the form $Y = X + b$

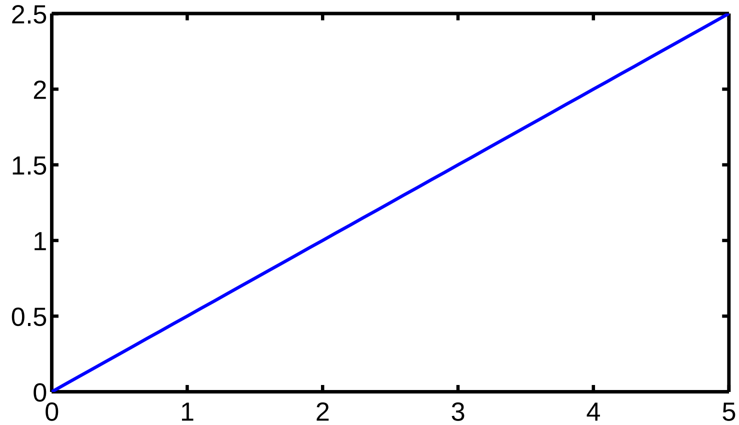
   - Then $F_Y(y)$
   - Thus $F_Y(y) =$ and $f_Y(y) =$
   - Note that the exponential density $F_X(x)$ is first nonzero at $x = 0$, so $F_Y(y)$ is first nonzero at $y - b = 0 \Rightarrow y = b$
   - Example: $b = 2$



  – So, adding a constant just ____________ the density

2. Functions of the form $Y = aX$, where $0 < a < 1$

   Let's start with a graphical approach:
   Example: $a = 0.5$

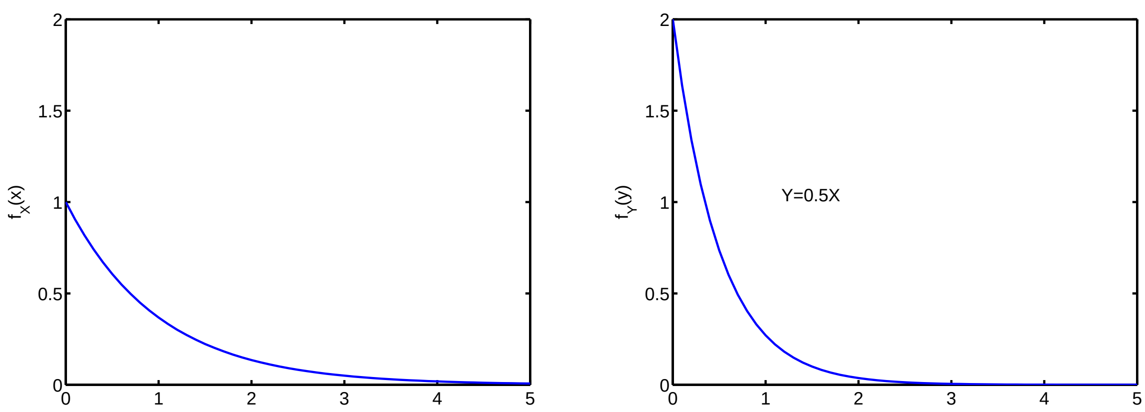


   So the probability is concentrated over a smaller range
   $\Rightarrow$ the density is ____________
   $F_Y(y) =$
   Thus $F_Y(y) =$ and $f_Y(y) =$
   Since $a < 1$, then $1/a > 1$

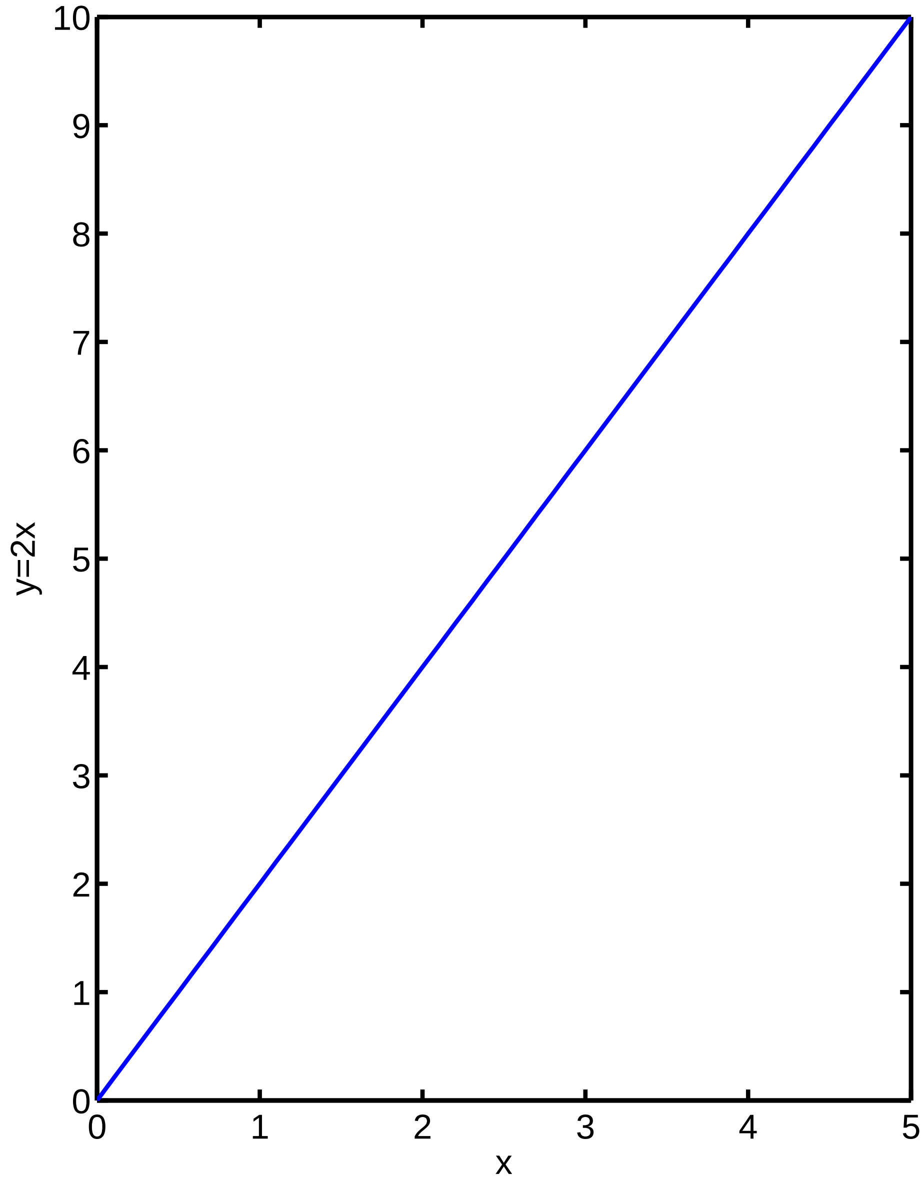
   The overall effect is that the density is ____________

   Example: $a = 0.5$

   $f_Y(y) = 2e^{-2y}$ (Y is also exponential)



3. Functions of the form $Y = aX$, where $a > 1$
   Again, start with a graphical approach:
   Example: $a = 2$



So the probability is now spread out over a larger range

$\Rightarrow$ the density is ______________

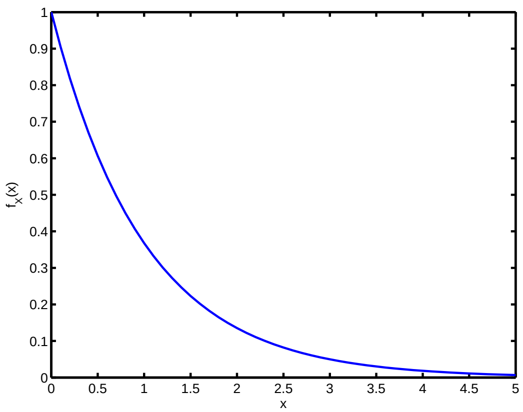
As before, $F_Y(y) = F_X(y/a)$ and $f_Y(y) = \frac{1}{a} f_X(y/a)$

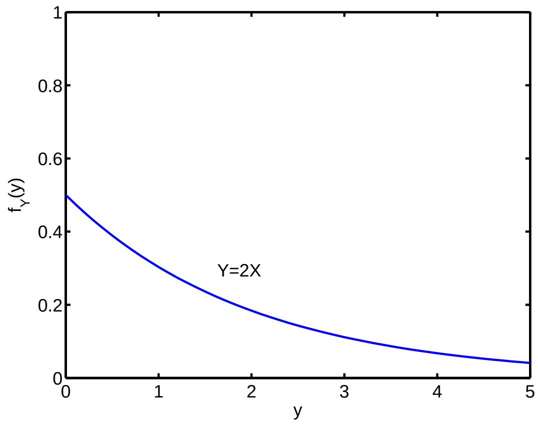
Since $a > 1$, then $1/a < 1$

The overall effect is that the density is ____________

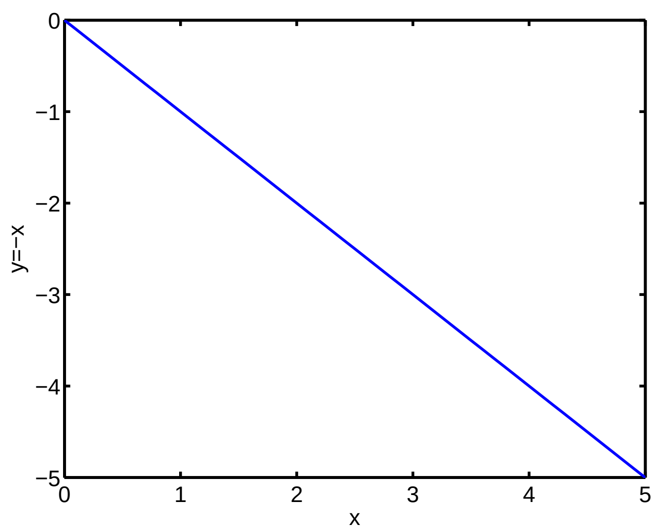
Example: $a = 2$

$f_Y(y) = 0.5e^{-y/2}$ (Y is still exponential)





4. Functions of the form $Y = -X$
   Start with the graphical approach:



So the probability density at every point x is move to every point $-x$. The density is ____________

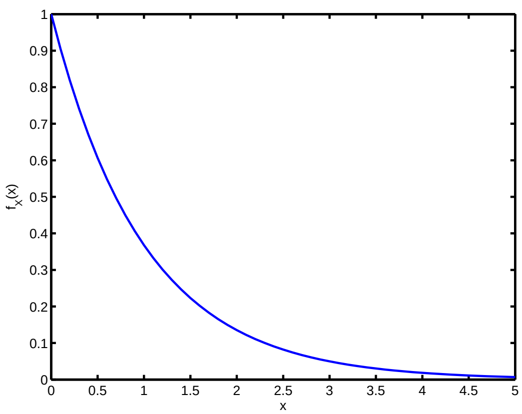
To verify this, let's check the mathematics:

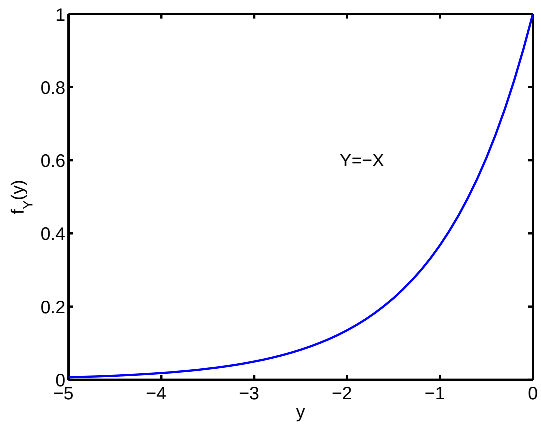
$F_Y(y) =$

Thus, $F_Y(y) =$ and $f_Y(y) =$

Example:$Y = -X$

$f_Y(y) = f_X(-y) = e^y u(-y)$





5. Functions of the form $Y = aX$, $a < 0$

   This case combines the previous cases: the probability density is flipped and stretched or compressed

   Mathematically,
   $F_Y(y) =$
   Thus $F_Y(y) =$ and $f_Y(y) =$

   Note that since $a < 0$, the density is still non-negative everywhere.
   Comparing with the previous cases $Y = aX$ for $a > 0$, we can give a single formula for the density as

   $$f_Y(y) =$$

6. General transformations $Y = g(X)$

   – Since any continuous function can be modeled as a series of infinitesimally small linear pieces, the previous types of changes to the density represent everything that can happen:

   – The density can be:

     1.

     2.

3.

4.

- Note that for general functions, all of these may apply at different points of the function.