EEL 5544 Noise in Linear Systems Lecture 1

What is a random experiment? **RANDOM EXPERIMENTS:**

Q:What do we mean by random?

Output is unpredictable in some sense.

DEFN A *random experiment* is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.

A *random experiment* is specified by:

1.

2.

DEFN An *outcome* of a random experiment is a result that ________________ ____.

DEFN The set of all possible outcomes for a random experiment is called the ___________________________________ and is denoted by S.

- **EX: Tossing a coin**

  A coin (heads on one side, tails on the other) is tossed one time, and the side that is face up is recorded.

  - *Q: What are the outcomes?*

- **EX: Rolling a 6-sided die**

  A 6-sided die is rolled, and the number on the top face is observed.

  - *Q:What are the outcomes?*

  A 6-sided die is rolled, and whether the top face is even or odd is observed.

- *Q:What are the outcomes?*

If the outcome of an experiment is random, how can we apply quantitative techniques to it?

This is the theory of probability.

People have tried to develop probability through a variety of approaches, which are discussed in the book in Section 1.2:

1. Probability as Intuition
2. Probability as the Ratio of Favorable to Unfavorable Outcomes (Classical Theory)
3. Probability as a Measure of Frequency of Occurrence
4. Probability Based on Axiomatic Theory

Probability as the Ratio of Favorable to Unfavorable Outcomes (Classical Theory)

- Some experiments are said to be "fair."
  - For a fair coin or a fair die toss, the outcomes are *equally likely*.
  - Equally likely outcomes are common in many situations.
- Problems involving a finite number of equally likely outcomes can be solved through the mathematics of counting, which is called *combinatorial analysis*.
- Given an experiment with equally likely outcomes, we can determine the probabilities easily.
- First, we need a little more knowledge about probabilities of an experiment with a finite number of outcomes:
  - An outcome or set of outcomes with probability 1 (one) is

    ____.
  - An outcome or set of outcomes with probability 0 (zero) is

    ____.
  - If you find a probability for a question on one of on my tests and your answer is < 0 or > 1, then you are certain to lose a lot of points.
- Using the first two rules, we can find the exact probabilities of individual outcomes for experiments with a finite number of equally likely outcomes.

EX: Tossing a fair coin

Let $p_H = \text{Prob}\{\text{heads}\}$, $p_T = \text{Prob}\{\text{tails}\}$

Then $p_H + p_T = 1$
(the probability that something occurs is 1!)

Since, $p_H = p_T$, $p_H = p_T = \frac{1}{2}$.
EX: Rolling a fair 6-sided die
Let $p_i = \text{Prob}\{\text{top face is } i\}$.Then

$$\begin{aligned} \sum_{i=1}^{6} p_i &= 1 \\ \Rightarrow 6p_1 &= 1 \\ \Rightarrow p_1 &= \frac{1}{6} \end{aligned}$$

So, $p_i = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

- We can use the probabilities of the outcomes to find probabilities that involve multiple outcomes:

EX: Rolling a fair 6-sided die

What is $\text{Prob}\{1 \text{ or } 2\}$?

$$\begin{aligned} \text{Prob}\{1 \text{ or } 2\} &= \text{Prob}\{1\} + \text{Prob}\{2\} \\ &= \frac{1}{6} + \frac{1}{6} \\ &= \frac{1}{3} \end{aligned}$$

The basic principle of counting
If two experiments are to be performed, where experiment 1 has m outcomes and experiment 2 has n outcomes for each outcome of experiment 1, then the combined experiment has ____________ outcomes.

We can use equally-likely outcomes on repeated trials, but we have to be careful:
EX: Rolling a fair 6-sided dice twice

Suppose we roll the die two times. What is the prob. of at least one 1?
On board.

What are some problems with defining probability in this way?

1. Requires equally likely outcomes – thus it only applies to a small set of random phenomena

2. Can only deal with a finite number of outcomes– this again limits its usefulness

Probability as a Measure of Frequency of Occurrence

- Consider a random experiment that has K possible outcomes, $K < \infty$
- Let $N_k(n) =$ the number of times the outcome is k
- Then we can tabulate $N_k(n)$ for various values of k and n
- We can reduce the dependence on n by dividing $N(k)$ by n to find out "how often did k occur".

RELATIVE FREQUENCY AND STATISTICAL REGULARITY

DEFN The *relative frequency* of outcome k of a random experiment is

- Observation: In our previous experiments, as n gets large, $f_k(n)$ converges to some constant value.

DEFN An experiment possesses *statistical regularity* if

DEFN For experiments with statistical regularity as defined above, p_k is called the

____.

PROPERTIES OF RELATIVE FREQUENCY

- Note that

  $$0 \leq N_k(n) \leq n, \forall k$$

  because $N_k(n)$ is just the # of times outcome k occurs in n trials.

  Dividing by n yields

  $$0 \leq \frac{N_k(n)}{n} = f_k(n) \leq 1,\ \forall k = 1, \ldots, K \tag{1}$$

- If $1, 2, ..., K$ are all of the possible outcomes, then

  $$\sum_{k=1}^{K} N_k(n) = n.$$

  Again, dividing by n yields

  $$\sum_{k=1}^{K} f_k(n) = 1. \tag{2}$$

  Consider the event E that an even number occurs.
  What can we say about the number of times E is observed in n trials?

  $$N_E(n) =$$

  What have we assumed in developing this equation?

  Then, dividing by n,

  $$\begin{aligned} f_E(n) &= \frac{N_E(n)}{n} = \\ &= \end{aligned}$$

- **General property:** If A and B are 2 events that cannot occur simultaneously, and C is the event that either A or B occurs, then

  $$f_C(n) = \tag{3}$$

- In some sense, we can define the probability of an event as its long-term relative frequency.

  What are some problems with this definition?

  1. It is not clear when and in what sense the limit exists.
  2. It is not possible to perform an experiment an infinite number of times, so the probabilities can never be known exactly.
  3. We cannot use this definition if the experiment cannot be repeated.

We need a *mathematical model of probability* that is not based on a particular application or interpretation.
However, any such model should

1. be useful for solving real problems
2. agree with our interpretation of probability as relative frequency
3. agree with our intuition (*where appropriate!*)