EEL 5544 Lecture 2

PROBABILITY SPACES

- We define a probability space as a mathematical construction containing three elements.We say that a probability space is a *triple*: $(\Omega, \mathcal{F}, P)$.
- The elements of a probability space for a random experiment are:

1. **DEFN** The *sample space*, denoted by S, is the __________ of ______________________________.

The sample space is also known as the universal set or reference set.

- Sample spaces come in two basic varieties: discrete or continuous

DEFN A *discrete set* is either ____________________ or ____.

DEFN A set is *countably infinite* if it can be put into one-to-one correspondence with the integers.

Examples

DEFN A *continuous set* is not countable.

DEFN An *interval* is a contiguous subset of the real line. If a and $b > a$ are in an interval I, then if $a \leq x \leq b$, $x \in I$.

- Intervals can be either open, closed, or half-open:
  - A closed interval $[a, b]$ contains the endpoints a and b.
  - An open interval (a, b) does not contain the endpoints a and b.
  - An interval can be half-open, such as $(a, b]$, which does not contain a, or $[a, b)$, which does not contain b.

- Intervals can also be either finite, infinite, or partially infinite.

Example of continuous sets

- We wish to ask questions about the probabilities of not only the outcomes but also **combinations of the outcomes**. For instance, if we roll a six-sided die and record the number on the top face, we may still ask questions like:
  - What is the probability that the result is even?
  - What is the probability that the result is ≤ 2?

DEFN *Events* are combinations of outcomes to which we assign probability.

Examples based on previous examples of sample spaces:

(a) Roll a fair 6-sided die and note the number on the top face.
Let $L_4 =$ the event that the result is less than or equal to 4
Express L *as a set of outcomes*

(b) Roll a fair 6-sided die and determine whether the number on the top face is even or odd.

Let $E =$ even outcome, $O =$ odd outcome
List all events.

(c) Toss a coin 3 times and note the sequence of outcomes.
Let H=heads, T=tails
Let A_1= event that heads occurs on first toss
Express A_1 as a set of outcomes.

(d) Toss a coin 3 times and note the number of heads.
Let O = odd number of heads occurs

Express O as a set of outcomes.

2. **DEFN** $\mathcal{F}$ is the *event class*, which is a collection of all events to which we assign probability.

- If Ω is discrete, then $\mathcal{F}$ can be taken to be the *power set* of Ω, which is the set of every subset of Ω
- If Ω is continuous, then weird things can happen if we consider every subset of Ω. For instance, if Ω itself has measure 1, we can construct a new set Ω' that consists only of subsets of Ω that has measure 2!
- The solution is to not assign a measure (i.e., probability) to some subsets of Ω, and therefore these things cannot be events.

DEFN A collection of subsets of Ω forms a *field* under the binary operations $\cup$ and $\cap$ if

(a) $\emptyset \in \mathcal{M}$ and $\Omega \in \mathcal{M}$
(b) If $E \in \mathcal{M}$ and $F \in \mathcal{M}$, then $E \cup F \in \mathcal{M}$ and $E \cap F \in \mathcal{M}$
(c) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.

DEFN A field $\mathcal{M}$ forms a σ-*algebra* or σ-field if for any $E_1, E_2, \ldots \in \mathcal{M}$

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}. \tag{2}$$

- Note that by property (c) of fields (2) can be interchanged with

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}. \tag{3}$$

- We require that the event class $\mathcal{F}$ be a σ-algebra.
- For discrete sets, the set of all subset of Ω is a σ-algebra.
- For continuous sets, there may be more than one possible σ-algebra possible.
- In this class, we only concern ourselves with the Borel field. On the real line ($\mathbb{R}$), the Borel field consists of all unions and intersections of intervals of $\mathbb{R}$.

3. **DEFN** The *probability measure*, denoted by P is a ______________ that maps all members of ________ onto ______________________________.

Axioms of Probability

- We specify a minimal set of rules that P must obey:

I.

II.

III.

Corollaries

- Properties of P that can be dreived from the axioms and the mathematical structure of $\mathcal{A}$:

(a) $P(A^c) = 1 - P(A)$

(b) $P(A) \leq 1$

(c) $P(\emptyset) = 0$

(d) If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive, then

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

(e) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof and example on board.

(f)

$$\begin{aligned} P\left(\bigcup_{k=1}^{n} A_k\right) &= \sum_{k=1}^{n} P\left(A_j\right) - \sum_{j<k} P\left(A_j \cap A_k\right) + \cdots \\ &\quad + (-1)^{(n+1)} P\left(A_1 \cap A_2 \cap \cdots \cap A_n\right) \end{aligned}$$

Add all single events, subtract off all intersections of pairs of events, add in all intersections of 3 events, ...

Proof is by induction.

(g) If $A \subset B$, then $P(A) \leq P(B)$.

Proof on board.