EEL 5544 Noise in Linear Systems Lecture 36

SPECIAL CASE: WSS GAUSSIAN RPS

- For Gaussian random processes, the n-dimensional distribution functions are completely specified by the __________ and __________________________
- If $X(t)$ is Gaussian and WSS, the means and autocovariances are shift-invariant
- Thus, if a random process is Gaussian and WSS, it is

  ____

MULTIPLE RANDOM PROCESSES

DEFN The *cross-correlation function* for two second-order random processes $X(t)$ and $Y(t)$ is

$$R_{XY}(t,s) = E[X(t)Y(s)]$$

DEFN The *cross-covariance function* for two second-order random processes $X(t)$ and $Y(t)$ is

$$C_{XY}(t,s) = E\left\{[X(t) - \mu_X(t)]\,[Y(s) - \mu_Y(s)]\right\}$$

Note that

$$C_{XY}(t,s) = R_{XY}(t,s) - \mu_X(t)\mu_Y(s)$$

DEFN Two second-order random processes $X(t)$ and $Y(t)$ are uncorrelated if

$$C_{XY}(t,s) \;=$$

$$\Rightarrow R_{XY}(t,s) \;=$$

DEFN Two random processes $X(t)$ and $Y(t)$ are *statistically independent* if for each $n \in \mathbb{Z}^+$ and each $t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n \in \mathbb{T}$,

$$[X(t_1)\,X(t_2)\ \ldots\ X(t_n)] \text{ and } [Y(t_1)\,Y(t_2)\ \ldots\ Y(t_n)]$$

are independent random vectors.

- Note that, as with random variables and random vectors, for random processes, s.i. $\Rightarrow$ uncorrelated, but uncorrelated does not necessarily imply s.i.
- For jointly Gaussian RPs, s.i. $\Leftrightarrow$ uncorrelated

DEFN Two RPs $X(t)$ and $Y(t)$ are *jointly wide-sense stationary* if

- jointly WSS is a stronger condition than each RP being WSS
- for jointly WSS RPs,

$$R_{X,Y}(t,s) = R_{X,Y}(t-s) = R_{X,Y}(\tau).$$

- If $X(t)$, $Y(s)$ are uncorrelated and $X(t)$ and $Y(t)$ are WSS, then

$$C_{X,Y}(t,s) = 0 \Rightarrow R_{X,Y}(t,x) = \mu_X \mu_Y$$

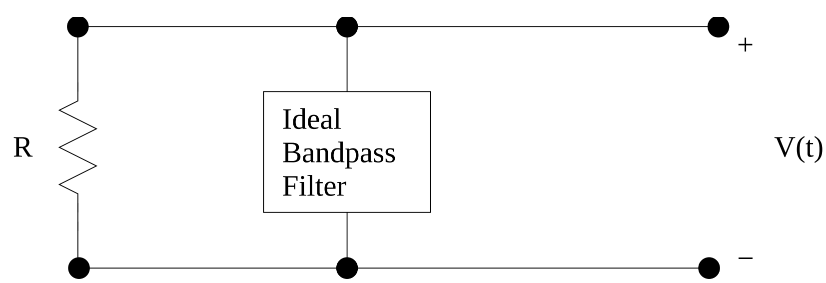
$\Rightarrow R_{X,Y}(t,s)$ is a constant that does not depend on either t or s

$\therefore$ uncorrelated RPs that are individually WSS are also jointly WSS

Examples

White Noise

- For second-order RPs, $E[X^2(t)] < \infty, \forall t \in \mathbf{T}$.
- White noise is a mathematical idealization of the thermal noise process
- Consider thermal noise measurement system:



- measurement shows power $4kTRB$ in each frequency band B

$\Rightarrow$ infinite power

1. model not valid for extremely high freqs.
2. no system has infinite bandwidth

$\therefore$ we assume that thermal noise can be modeled as white noise, with equal power at all frequencies

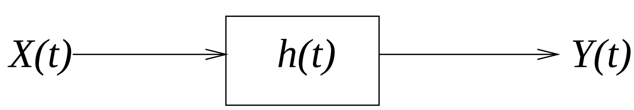
- autocorrelation fcn for white noise is

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau), \tag{1}$$

where the value of $N_0/2$ is called the (two-sided) noise power spectral density

- We use white Gaussian noise (WGN) to model thermal noise. For engineering purposes, we treat WGN as a zero-mean Gaussian random process with autocorrelation function given by (1)
- If $X(t)$ is WGN, then $E[X^2(t)] = R_X(0) = \infty$, so the mean does not necessarily exist.
- A more careful definition of WGN is $X(t)$ is a WGN process if, when input to an LTI filter with $h(t) < \infty$ for all t, then $Y(t) = [h * X](t)$ is a zero-mean Gaussian random process.

TIME-INVARIANT FILTERING



$$\begin{aligned} & Y = X * h = h * X \\ \Rightarrow \quad & Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau \end{aligned}$$

- Stability (BIBO = bounded-input, bounded-output):
Necessary & sufficient condition is $\exists M_h < \infty \ni$

$$\int_{-\infty}^{\infty} |h(t)|\, dt \leq M_h.$$

Then $|X(t)| < \infty \forall t \Rightarrow |Y(t)| < \infty \forall t$.

Calculation of Mean and Autocorrelation Functions for WSS RPs

- Suppose:
  - $X(t)$ is WSS 2nd-order RP with mean μ_X and autocorrelation $R_X(\tau)$
  - $X(t)$ input to BIBO LTI system with impulse response $h(t)$.
  - $Y(t)$ denotes the output process.

  Then

$$\begin{aligned}\mu_Y(t) &= E\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} E\left[X(\tau)\right] h(t-\tau)d\tau \\ &= \mu_X \int_{-\infty}^{\infty} h(t-\tau)d\tau \\ &= \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau\end{aligned}$$

$\Rightarrow$ The mean of $Y(t)$ is a constant.

  The autocorrelation function for $Y(t)$ is

$$\begin{aligned}R_Y(t_1,t_2) &= E\left\{Y(t_1)Y(t_2)\right\} \\ &= E\left\{\int_{-\infty}^{\infty} X(t_1-u)h(u)du \int_{-\infty}^{\infty} X(t_2-v)h(v)dv\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\Big[X(t_1-u)X(t_2-v)\Big]h(u)h(v)du\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1-t_2-u+v)h(u)h(v)du\,dv\end{aligned}$$

- Note that R_Y only depends on t_1 and t_2 through $t_1 - t_2$.

This with the fact that the mean of $Y(t)$ is constant $\Rightarrow Y(t)$ is also WSS.

- The expression for R_Y can be further simplified as

$$\begin{aligned}R_Y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-u+v)h(u)h(v)dudv \\ &= (h * \tilde{h} * R_X)(\tau), \qquad (2)\end{aligned}$$

  where $\tilde{h}(t) = h(-t)$ (the time-reverse of $h(t)$

- To make computation even simpler, it is often convenient to compute an intermediate function. Let

$$\begin{aligned} f(\tau) &= (h * \tilde{h})(\tau) \\ &= \int_{-\infty}^{\infty} h(t)h(t-\tau)d\tau. \end{aligned}$$

Then

$$R_Y(\tau) = (f * R_X)(\tau) \tag{3}$$

- Note that:
  - $h(t-\tau)$ is a time-shifted version of $h(t)$
  - typically computing $f(\tau)$ is easier than performing normal convolution
  - $f(\tau)$ is an even function

SPECIAL CASE: : WHITE GAUSSIAN NOISE (WGN)

- For WGN, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$
- Then using (3) and the sifting property of the delta function, $R_Y(\tau) = \frac{N_0}{2} f(\tau)$