EEL 5544 Noise in Linear Systems Lecture 37

Power in the Output Process

If only the power of the output process, it is not necessary to do the full convolution of f and R_x:

$$\begin{aligned} R_Y(0) &= \int_{-\infty}^{\infty} f(\tau) R_X(\tau) d\tau \\ &= 2\int_{0}^{\infty} f(\tau) R_X(\tau) d\tau. \end{aligned}$$

For WGN inputs,

$$R_Y(0) = \frac{N_0}{2} f(0),$$

where

$$\begin{aligned} f(0) &= \int_{-\infty}^{\infty} h(u)h(u-0)du \\ &= \int_{-\infty}^{\infty} h^2(t)dt = \|h\|^2, \end{aligned}$$

so

$$E[Y^2(t)] = \frac{N_0}{2}\|h\|^2$$

Examples

Calculation of Mean and Autocorrelation Functions for WSS RPs in LTI Systems

- The above approach is an effective way to find $R_Y(\tau)$, but it does not identify whether $X(t)$ and $Y(t)$ are jointly WSS
- Consider an alternative approach to finding $R_Y(\tau)$:
- The cross-correlation function $R_{YX}(t+\tau, t)$ is

$$\begin{aligned} R_{YX}(t+\tau,t) &= E[Y(t+\tau)X(t)] \\ &= E[Y(t)X(t-\tau)] \\ &= E\left[\int_{-\infty}^{\infty} h(u)X(t-u)du X(t-\tau)\right] \\ &= \int_{-\infty}^{\infty} h(u)E\left[X(t-u)X(t-\tau)\right]du \\ &= \int_{-\infty}^{\infty} h(u)R_X(\tau-\alpha)du \\ &= (h * R_X)(\tau) \end{aligned}$$

- Then from our previous work, $R_Y(\tau) = (\tilde{h} * R_{YX})(\tau)$
- Note that $R_{YX}(t+\tau, t) = R_{YX}(\tau)$ implies that $X(t)$ and $Y(t)$ are also jointly WSS

POWER SPECTRAL DENSITY

DEFN If $X(t)$ is a WSS RP, then the ________________________ ________________________ of $X(t)$ is

$$S_X(f) = \mathcal{F}\left\{R_X(\tau)\right\}.$$

PROPERTIES OF PSD FOR REAL $X(t)$

1. $S_X(f) = S_X(-f), \forall f$ (even)
2. $S_X(f) = [S_X(f)]^*, \forall f$ (real)
3. $S_X(f) \geq 0, \forall f$ (non-negative)
4. If

$$\int_{-\infty}^{\infty} |R_X(\tau)|\, d\tau < \infty$$

   then $S_X(f)$ is a continous function of f.

DEFN If $X(t)$ and $Y(t)$ are jointly WSS RPs, then the ____________________ ____________________________________ of $X(t)$ is

$$S_X(f) = \mathcal{F}\left\{R_{XY}(\tau)\right\}.$$

- Note that the cross-PSD is generally complex-valued and does not satisfy the properties of a PSD

PSD OF WHITE GAUSSIAN NOISE

- For WGN, $R_X(\tau) = (N_0/2)\delta(\tau)$
- Then $S_X(\omega) = \mathcal{F}\{R_X(\tau)\} = N_0/2$

- I.e., white noise has equal power density at every frequency
  (like white light, which has equal power at every wavelength)
- The origin of the factor of (1/2) comes from the two-sided nature of the Fourier transform.If we have an ideal filter of bandwidth B such that $H(f) = 1$, $f_0 - B/2 \leq |f| \leq f_0 + B/2$, then we will see that the power at the output of the filter is BN_0
- The PSD can be an easy way to characterize the output process of a BIBO LTI system:

  Let $H(f) = \mathcal{F}\{h(t)\}$, then

$$\begin{aligned} S_Y(f) &= \mathcal{F}\left\{h * \tilde{h} * R_X\right\} \\ &= H(f)H^*(f)S_X(f) \\ &= |H(f)|^2\, S_X(f). \end{aligned}$$

- The output power is given by

$$\begin{aligned} E[Y^2(t)] &= R_Y(0) \\ &= \int_{-\infty}^{\infty} |H(f)|^2\, S_X(f) df. \end{aligned}$$

<u>SUFFICIENT CONDITION FOR A FUNCTION TO BE A PSD</u>

- Suppose $F(\omega)$ is a real-valued function such that

$$F(\omega) \geq 0 \ \forall \omega \tag{1}$$

- Then let $H(\omega) = \sqrt{F(\omega)}$ and input WGN with PSD 1 to a filter with frequency response $H(\omega)$
- Then the output $Y(t)$ has PSD $S_Y(\omega) = F(\omega)$
- Thus any function that satisfies (1) is a valid PSD
- Moreover, the condition that an autocorrelation function be positive semidefinite is equivalent to (1)
- Thus, a *necessary* and *sufficient condition* for a function $f(\tau)$ to be a valid autocorrelation function is for $f(\tau)$ to be positive semidefinite
- It is easier to determine if a function is a valid autocorrelation function by transforming it into the frequency domain