EEL 5544 Noise in Linear Systems Lecture 3

STATISTICAL INDEPENDENCE

DEFN Two events $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are ______________________________, abbreviated ______, if and only if ________________________.

- Later we will show that if events are statistically independent, then knowledge that one event occurs does not affect the probability that the other event occurs.
- Events that arise from completely separate random phenomena will be statistically independent.
- **It is important to note that statistical independence is a ________________________________.**

  Claim: If A and B are s.i. events, then the following pairs of events are also s.i.:

  - A and $\overline{B}$
  - $\overline{A}$ and B
  - $\overline{A}$ and $\overline{B}$

  Proof: It is sufficient to consider the first case, i.e. show that if A and B are s.i. events then A and $\overline{B}$ are also s.i. events. The other cases follow directly.

  Note that $A = (A \cap B) \cup (A \cap \overline{B})$ and $A = (A \cap B) \cap (A \cap \overline{B}) = \emptyset$. Thus

$$
\begin{aligned}
P(A) &= P[(A \cap B) \cup (A \cap \overline{B})] \\
&= P(A \cap B) + P(A \cap \overline{B}) \\
&= P(A)P(B) + P(A \cap \overline{B})
\end{aligned}
$$

  Thus

$$
\begin{aligned}
P(A \cap \overline{B}) &= P(A) - P(A)P(B) \\
&= P(A)[1 - P(B)] \\
&= P(A)P(\overline{B}) \quad \square
\end{aligned}
$$

- For N events to be s.i., require:

$$
\begin{aligned}
P(A_i \cap A_k) &= P(A_i)P(A_k) \\
P(A_i \cap A_k \cap A_m) &= P(A_i)P(A_k)P(A_m) \\
&\vdots \\
P(A_1 \cap A_1 \cap \cdots \cap A_N) &= P(A_1)P(A_2)\cdots P(A_N)
\end{aligned}
$$

  (It is not sufficient to have $P(A_i \cap A_k) = P(A_i)P(A_k), \forall i \neq k$.)

Weaker Condition: Pairwise Statistical Independence

DEFN N events $A_1, A_2, \ldots, A_N \in \mathcal{A}$ are *pairwise statistically independent* if and only if $P(A_i \cap A_j) = P(A_i)P(A_j), \forall i \neq j$.

Examples of Applying Statistical Independence

On other sheets

RELATION BETWEEN STATISTICALLY INDEPENDENT AND MUTUALLY EXCLUSIVE EVENTS

- One common mistake of students learning probability is to confuse statistical independence with mutual exclusiveness
- Remember:
  - Mutual exclusiveness is a __________________
  - Statistical independence is a ________________________________
- How does mutual exclusiveness relate to statistical independence?
  1. m.e. $\Rightarrow$ s.i. ?
  2. s.i. $\Rightarrow$ m.e. ?
  3. two events cannot be both s.i. and m.e., except in some degenerate cases
  4. none of the above
- Perhaps surprisingly, __________ is true

  Consider the following:

COMBINATORICS

- **Basic principle of counting:** The number of distinct, ordered k-tuples $(x_1, x_2, \ldots, x_k)$ that are possible if x_i is an element of a set with n_i distinct members is

____

SPECIAL CASES

1. **Sampling with repacement and with ordering**

   Problem: Choose k objects from a set A with replacement (after each choice, the object is returned to the set). The ordered series of results is noted.

   Result is k-tuple: $(x_1, x_2, \ldots, x_k)$,
   where $x_i \in A, \forall i = 1, 2, \ldots, k$.

   Thus, $n_1 = n_2 = \ldots = n_k = |A| \equiv n$.

   $\Rightarrow$ number of distinct ordered k-tuples is _____.

2. **Sampling w/out replacement & with ordering**

   Problem: Choose k objects from a set A without replacement (after each object is selected, it is removed from the set). Let $n = |A|$, where $k \leq n$

   Result is k-tuple: $(x_1, x_2, \ldots, x_k)$,
   where $x_i \in A, \forall i = 1, 2, \ldots, k$.

   Then $n_1 = n, n_2 = n - 1, \ldots, n_k = n - (k - 1)$.

   $\Rightarrow$ number of distinct ordered k-tuples is

3. **Permutations of n distinct objects**

   Problem: Find the # of permutations of a set of n objects. (The # of permutations is the # of possible orderings.)

   Result is n-tuple.

   The # of orderings is the # of ways we can take the objects one at a time from a set without replacement until no objects are left in the set.

   Then $n_1 = n, n_2 = n - 1, \ldots, n_{n-1} = 2, n_n = 1$.

⇒ # of permutations	=	# of distinct ordered n-tuples
	=	
	=	

Useful formula: For large n,

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

MATLAB TECHNIQUE: To determine $n!$ in MATLAB use `gamma(n+1)`

4. **Sample w/out Replacement & w/out Order**

   Question: How many ways are there to pick k objects from a set of n objects without replacement and without regard to order?

   Equivalently: "How many subsets of size k (called a combination of size k) are there from a set of size n?"

   First determine # of ordered subsets:

   Now, how many different orders are there for any set of k objects?

   Let C_k^n = the number of combinations of size k from a set of n objects

$$C_k^n = \frac{\text{\# ordered subsets}}{\text{\# orderings}} =$$

DEFN The number of ways in which a set of size n can be partitioned into J subsets of size $k_1, k_2, \ldots, k_J$ is given by the *multinomial coefficient*

- The multinomial coefficient assumes that the subsets are ordered. If the subsets are not ordered, then divide by the number of permutations of subsets.
- **EX:** Suppose there are 72 students in the class, and I randomly divide them into groups of 3 and assign each group a different project. How many different arrangements of groups are there?

Using the multinomial coefficient, there are

$$\frac{72!}{(3!)^{24}} \approx 1.3 \times 10^{85}$$

Order matters because each group gets a different project.

- **EX:** Now suppose there are 72 students that work in groups of 3, where each group works on the same project.

  There are $24!$ ways to arrange the 24 groups, so the total number of ways to assign people to groups is

  $$\frac{72!}{(3!)^{24}\,(24!)} \approx 2.1 \times 10^{61} \text{ ordered groups}$$

- **EX. What is the probability that each number appears exactly twice in 12 rolls of 6-sided die**

  One possible way: $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6$

  – How many total ways can this occur?
  – What is the prob. of this occurring?

  On other sheets

5. **Sample with Replacement and w/out Ordering**

   Question: How many ways are there to choose k objects from a set of n objects without regard to order?

   - The total number of arrangements is

     $$\binom{k+n-1}{k}.$$

   - I will provide some details about this result on the course website

Conditional Probability: Consider an example

EX Defective computers in a lab

A computer lab contains

- two computer from manufacturer A, one of which is defective
- three computers from manufacturer B, two of which are defective

A user sits down at a computer at random. Let the properties of the computer he sits down at be denoted by a two letter code, where the first letter is the manufacturer and the second letter is D for a defective computer and N for a non-defective computer.

$$\Omega = \{AD, AN, BD, BD, BN\}$$

Let

- E_A be the event that the selected computer is from manufacturer A
- E_B be the event that the selected computer is from manufacturer B
- E_D be the event that the selected computer is defective

Find

$P(E_A) =$________	$P(E_B) =$________	$P(E_D) =$________

- Now, suppose that I select a computer and tell you its manufacturer. Does that influence the probability that the computer is defective?
- Ex: Suppose I tell you the computer is from manufacturer A. Then what is the prob. that it is defective?

  We denote this prob. as $P(E_D|E_A)$
  (means: *the conditional probability of event E_D given that event E_A occured*)

$$\Omega = \{AD, AN, BD, BD, BN\}$$

  Find

  - $P(E_D|E_B) =$________
  - $P(E_A|E_D) =$________
  - $P(E_B|E_D) =$________

Next time: we need a systematic way of determining probabilities given additional information about the experiment outcome.