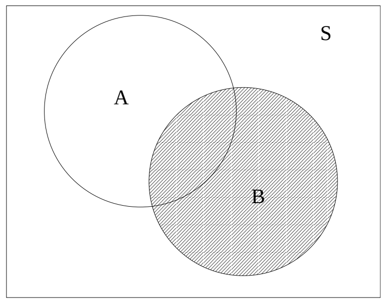
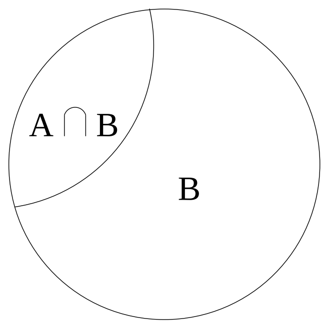
EEL 5544 Noise in Linear Systems Lecture 4

CONDITIONAL PROBABILTY

Consider the Venn diagram:



If we condition on B having occurred, then we can form the new Venn diagram:



This diagram suggests that if $A \cap B = \emptyset$ then if B occurs, A could not have occurred.

Similarly if $B \subset A$, then if B occurs, the diagram suggests that A must have occurred.

A definition of conditional probability that agrees with these and other observations is:

DEFN For $A \in \mathcal{A}$, $B \in \mathcal{A}$, the *conditional probability* of event A *given* that event B occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ for } P(B) > 0.$$

Claim: If $P(B) > 0$, the conditional probability $P(|B)$ forms a valid probability space on the original sample space
$(S, \mathcal{A}, P(\cdot|B))$

Check the axioms:

1.

$$P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

2. Given $A \in \mathcal{A}$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

and $P(A \cap B) \geq 0$, $P(B) \geq 0$

$$\Rightarrow P(A|B) \geq 0$$

3. If $A \subset \mathcal{A}$, $C \subset \mathcal{A}$, and $A \cap C = \emptyset$,

$$\begin{aligned} P(A \cup C|B) &= \frac{P\left[(A \cup C) \cap B\right]}{P[B]} \\ &= \frac{P\left[(A \cap B) \cup (C \cap B)\right]}{P[B]}. \end{aligned}$$

Note that $A \cap C = \emptyset \Rightarrow (A \cap B) \cap (C \cap B) = \emptyset$, so

$$\begin{aligned} P(A \cup C|B) &= \frac{P\left[A \cap B\right]}{P\left[B\right]} + \frac{P\left[C \cap B\right]}{P[B]} \\ &= P(A|B) + P(C|B) \end{aligned}$$

Check prev. example: $\Omega = \{AD, AN, BD, BD, BN\}$

$P(E_D|E_A) =$

$P(E_D|E_B) =$

$P(E_A|E_D) =$

$P(E_B|E_D) =$

Ex: Drawing two consecutive balls without replacement

An urn contains

- two black balls, numbered 1 and 2
- two white balls, numbered 3, 4 and 5

Two balls are drawn from the urn without replacement. What is the probability of getting a white ball on the second draw given that you got a white ball on the first draw?

Let W_i= event that white ball chosen on draw i

What is $P(W_2|W_1)$?

CHAIN RULE FOR EXPANDING INTERSECTIONS

$$\begin{aligned}
\text{Note that } P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
\Rightarrow P(A \cap B) &= P(A|B)P(B) \qquad (1) \\
\text{and } P(B|A) &= \frac{P(A \cap B)}{P(A)} \\
\Rightarrow P(A \cap B) &= P(B|A)P(A) \qquad (2)
\end{aligned}$$

- Eqns. (1) and (2) are chain rules for expanding the probability of the intersection of two events
- The chain rule can be easily generalized to more than two events

Ex: Intersection of 3 events

$$\begin{aligned}
P(A \cap B \cap C) &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)} \cdot P(C) \\
&= P(A|B \cap C)P(B|C)P(C)
\end{aligned}$$

RELATING CONDITIONAL AND UNCONDITIONAL PROBS

Which of the following statements are true?

(a) $P(A|B) \geq P(A)$

(b) $P(A|B) \leq P(A)$

(c) Not necessarily (a) or (b)

Partitions and Total Probability

DEFN A collection of sets $A_1, A_2, \ldots$ *partitions* the sample space S if and only if

$\{A_i\}$ is also said to be a *partition* of S.

1. Venn Diagram

2. Expressing an arbitrary set using a partition of S.

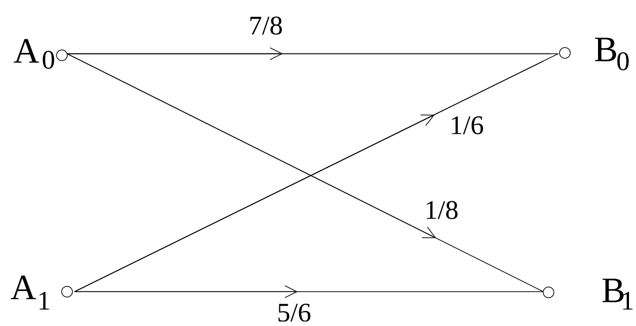
Law of Total Probability

If $\{A_i\}$ partitions S, then

Example

Binary Communication System

- A transmitter (Tx) sends A_0, A_1
  - A_0 is sent with prob. $P(A_0) = 2/5$
  - A_1 is sent with prob. $P(A_1) = 3/5$
- A receiver (Rx) processes the output of the channel into one of two values B_0, B_1
- The channel is a noisy channel that determines the probabilities of transitions between A_0, A_1 and B_0, B_1
  - The transitions are conditional probabilities $P(B_j|A_i)$ that can be specified on a *channel transition diagram* :



- The receiver must use a *decision rule* to determine from the output B_0 or B_1 whether the the symbol that was sent was most likely A_0 or A_1
- Let's start by considering 2 possible decision rules:
  1. Always decide A_1
  2. When receive B_0 decide A_0, when receive B_1 decide A_1
- **Problem:** Determine the probability of error for these two decision rules.