EEL 5544 Lecture 9

IMPORTANT RANDOM VARIABLES-CONT.

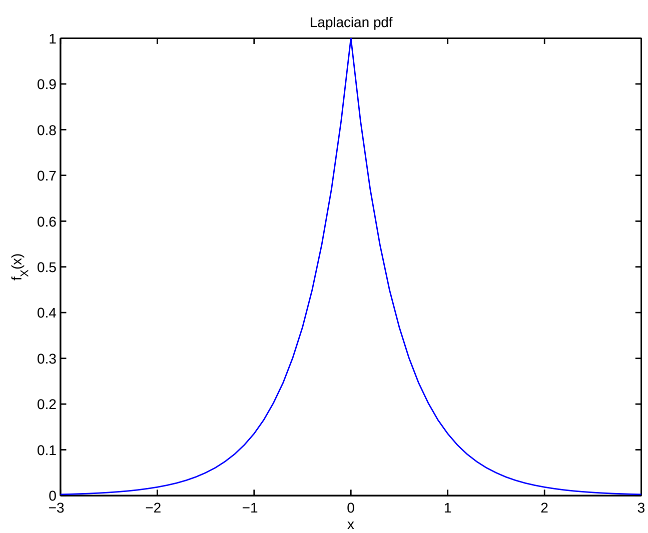
4. Laplacian RV

- The Laplacian random variable is often used to model the difference between correlated data sources
- For example, the differences between adjacent samples of speech, images, or video are often modeled using Laplacian random variables
- The density function for a Laplacian random variable is

$$f_X(x) = \frac{c}{2} \exp\{-c|x|\}, -\infty < x < \infty$$

where $c > 0$

- The pdf for a Laplacian RV with $c = 2$ is shown below



5. Chi-square RV

- Let X be a Gaussian random variable
- Then $Y = X^2$ is a chi-square random variable

- But in fact, chi-square random variables are much more general. For instance, if X_i, $i = 1, 2, \ldots, n$ are s.i. *Gaussian* random variables with identical variances σ^2, then

$$Y = \sum_{i=1}^{n} X_i^2 \tag{1}$$

  is a chi-square random variable with n degrees of freedom

- The chi-square random variable is usually classified as either *central* or *non-central*

- If in (1), all of the Gaussian random variables have mean $\mu = 0$, then Y is a central chi-square random variable

- The density of a central chi-square random variable is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sigma^n 2^{n/2}\Gamma(n/2)} y^{(n/2)-1} e^{-y/(2\sigma^2)}, & y \geq 0, \\ 0, & y < 0 \end{cases}$$

  where $\Gamma(p)$ is the gamma function defined as

$$\begin{cases} \Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, & p > 0 \\ \Gamma(p) = (p-1)!, & p \text{ an integer} > 0 \\ \Gamma(1/2) = \sqrt{\pi} \\ \Gamma(3/2) = \frac{1}{2}\sqrt{\pi} \end{cases}$$

- Note that for $n = 2$,

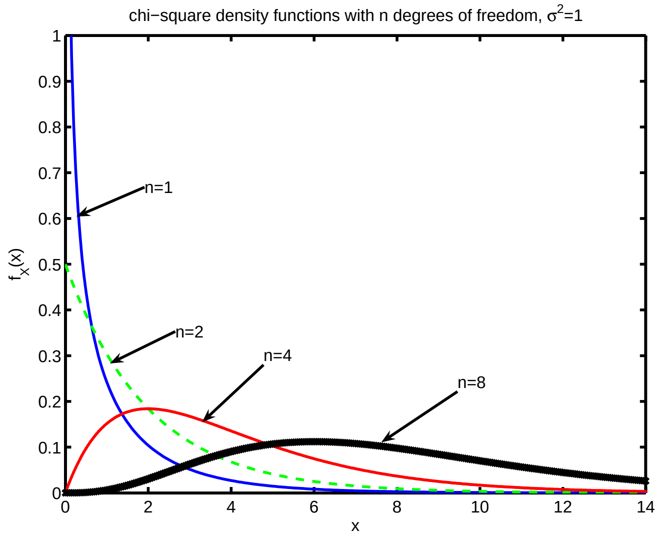
$$f_Y(y) = \begin{cases} \frac{1}{2\sigma^2} e^{-y/(2\sigma^2)}, & y \geq 0, \\ 0, & y < 0 \end{cases}$$

  which is an exponential density

- Thus, the sum of the squares of two s.i. zero-mean Gaussian RVs with equal variance is exponential!

- Note also that although the Gaussian random variables that make up the central chi-square RV have zero mean, the chi-square RV has mean > 0

- The pdf for chi-square RVs with 1,2, 4, or 8 degrees of freedom and $\sigma^2 = 1$ are shown below



- The CDF for a central chi-square RV can be found through repeated integration by parts to be

$$F_Y(y) = \begin{cases} 1 - e^{-y/(2\sigma^2)} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{y}{2\sigma^2}\right)^k, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

- The non-central chi-square random variable has more complicated pdfs and CDFs, and the CDF cannot be written in closed form

6. Rayleigh RV

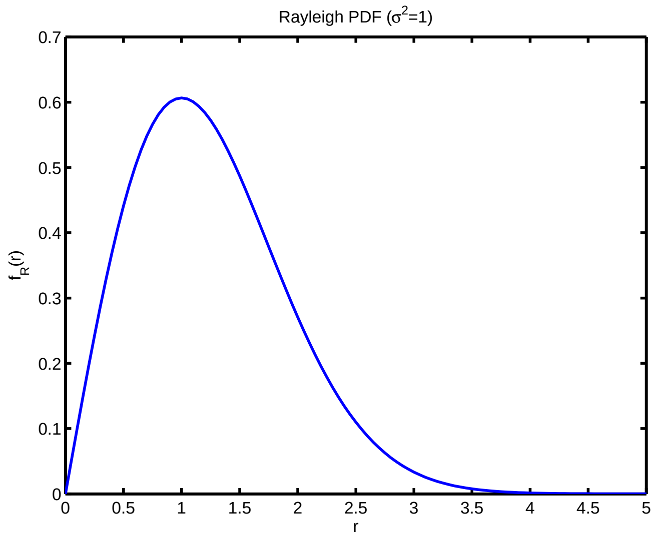
- Consider an exponential RV Y, which is equivalent to a central chi-square RV with 2 degrees of freedom, which is equivalent to the sum of the squares of independent, zero-mean Gaussian RVs with common variance

- Let $R = \sqrt{Y}$

- Then R is a Rayleigh RV with pdf

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

- The pdf for Rayleigh RV with $\sigma^2 = 1$ is shown below



- The CDF for a Rayleigh RV is

$$F_R(r) = \begin{cases} 1 - e^{-r^2/(2\sigma^2)}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

- The amplitude of a radio signal in a dense multipath environment is often modeled as a complex Gaussian random variable with independent real and imaginary parts. Thus, the amplitude of the waveform is a Rayleigh random variable.

7. <u>Lognormal RV</u>

- Consider a random variable L such that $X = \ln L$ is a Gaussian RV

- For instance in communications, we find that shadowing of communication signals by buildings and foliage has a Gaussian distribution when the signal loss is expressed in decibels

- Then L is a lognormal random variable with pdf given by

$$f_L(\ell) = \begin{cases} \frac{1}{\ell\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln \ell - \mu)^2}{2\sigma^2}\right\}, & \ell \geq 0 \\ 0, & \ell < 0 \end{cases}$$

CONDITIONAL DISTRIBUTIONS AND DENSITIES

- Given $(S, \mathcal{A}, P)$ and $X : S \rightarrow \mathbf{R}$ (a RV on S):

DEFN For an event $A \in \mathcal{A}$ with $P(A) \neq 0$, the *conditional distribution of X given A* is

$$f_X(x|A) \quad =$$

DEFN For an event $A \in \mathcal{A}$ with $P(A) \neq 0$, the *conditional density of X given A* is

$$f_X(x|A) \quad =$$

- Then $F_X(x|A)$ is a distribution function, and $f_X(x|A)$ is a density function.

  Example Suppose that the waiting time in minutes at Wendy's in the Reitz Union is an exponential random variable with parameter $\lambda = 1$. Given that you have already been waiting for two minutes, then what is

  (a) the conditional distribution for your total wait time?

  (b) the conditional distribution for your remaining wait time?

TOTAL PROBABILITY AND BAYE'S THEOREM

- If $A_1, A_2, \ldots, A_n$ form a partition of S, then

$$\begin{aligned} F_X(x) \quad &= \quad F_X(x|A_1)P(A_1) + F_X(x|A_2)P(A_2) \\ &\quad + \ldots + F_X(x|A_n)P(A_n) \end{aligned}$$

  and

$$f_X(x) = \sum_{i=1}^{n} f_X(x|A_i)P(A_i).$$

Point Conditioning:

- Suppose we want to evaluate the probability of an event given that $X = x$, where X is a continuous random variable.
- Clearly, $P(X = x) = 0$, so the previous definition of conditional prob. will not work.

$$\begin{aligned}
P(A|X=x) &= \lim_{\Delta x \to 0} P(A|x < X \leq x + \Delta x) \\
&= \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x|A) - F_X(x|A)}{F_X(x+\Delta x) - F_X(x)} P(A) \\
&= \lim_{\Delta x \to 0} \frac{\frac{F_X(x+\Delta x|A) - F_X(x|A)}{\Delta x}}{\frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}} P(A) \\
&= \frac{f_X(x|A)}{f_X(x)} P(A),
\end{aligned}$$

if $f_X(x|A)$ and $f_X(x)$ exist.

Implication of Point Conditioning

- Note that

$$P(A|X=x) = \frac{f_X(x|A)}{f_X(x)} P(A)$$

$$\Rightarrow \quad P(A|X=x) f_X(x) = f_X(x|A) P(A)$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} P(A|X=x) f_X(x) dx = \int_{-\infty}^{\infty} f_X(x|A) dx P(A)$$

$$\Rightarrow \quad P(A) = \int_{-\infty}^{\infty} P(A|X=x) f_X(x) dx$$

(Continuous Version of Law of Total Probability)

- Once we have the Law of Total Probability, it is easy to derive the **Continuous Version of Baye's Theorem**:

$$\begin{aligned}
f_X(x|A) &= \frac{P(A|X=x)}{P(A)} f_X(x) \\
&= \frac{P(A|X=x) f_X(x)}{\int_{-\infty}^{\infty} P(A|X=t) f_X(t) dt}.
\end{aligned}$$

Example on board.