1.3.1 CONTINUOUS-TIME FOURIER SERIES (CTFS)

Spectral representation for *periodic* CT signals

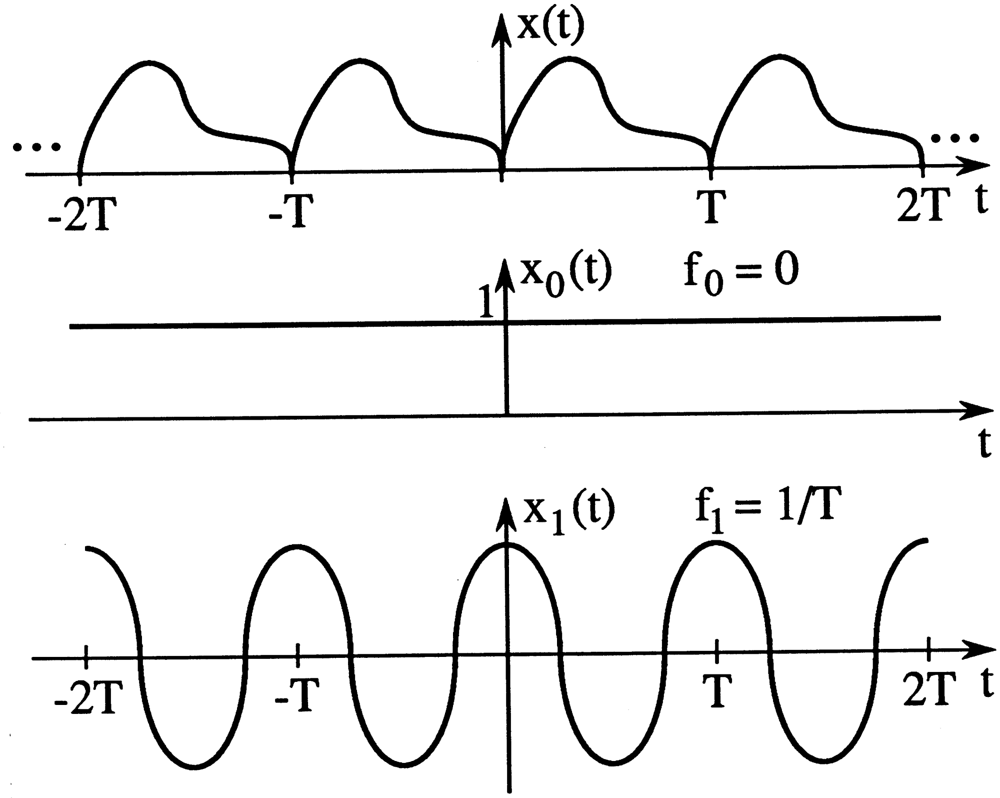
Assume $x(t) = x(t + nT)$

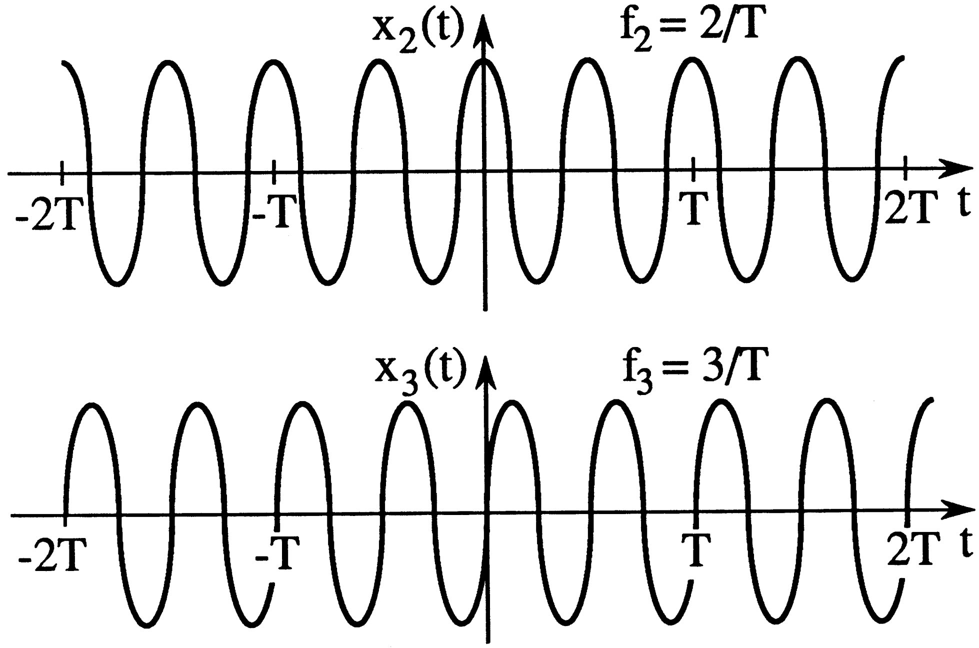
We seek a representation for x(t) of the form

$$x(t) = \frac{1}{T} \sum_{k=0}^{N-1} A_k x_k(t) \tag{1a}$$

$$x_k(t) = \cos(2\pi f_k t + \theta_k) \tag{1b}$$

Since x(t) is periodic with period T, we only use frequency components that are periodic with this period:





How do we choose amplitude A_k and phase θ_k of each component?

First, put into more convenient form

$k \neq 0$:

$$
\begin{aligned}
A_k x_k(t) &= A_k \cos(2\pi kt/T + \theta_k) \\
&= \frac{A_k}{2}\, e^{j\theta_k}\, e^{j2\pi kt/T} + \frac{A_k}{2}\, e^{-j\theta_k}\, e^{-j2\pi kt/T} \\
&= X_k\, e^{j2\pi kt/T} + X_{-k}\, e^{-j2\pi kt/T}
\end{aligned}
$$

$k = 0$:

$$
\begin{aligned}
A_0 x_0(t) &= A_0 \cos(\theta_0) \\
&= X_0
\end{aligned}
$$

Given a fixed signal x(t), we don't know at the outset whether an *exact* representation of the form given by Eq. (1) even exists.

However, we can always *approximate* x(t) by an expression of this form. So we consider

$$\hat{x}(t) = \frac{1}{T} \sum_{k=-(N-1)}^{N-1} X_k \, e^{j2\pi kt/T}$$

We want to choose the coefficients X_k so that $\hat{x}(t)$ is a good approximation to $x(t)$.

Define $e(t) = \hat{x}(t) - x(t)$.

We would like $e(t)$ to be "small" in some sense.

Recall $$P_e = \frac{1}{T} \int_{-T/2}^{T/2} |e(t)|^2 dt$$

$$P_e = \frac{1}{T} \int_{-T/2}^{T/2} | \hat{x}(t) - x(t) |^2 dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left| \frac{1}{T} \sum_{k=-(N-1)}^{N-1} X_k \, e^{j2\pi kt/T} - x(t) \right|^2 dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left[\frac{1}{T} \sum_{k=-(N-1)}^{N-1} X_k \, e^{j2\pi kt/T} - x(t) \right]$$

$$\times \left[\frac{1}{T} \sum_{\ell=-(N-1)}^{N-1} X_\ell^* \, e^{j2\pi \ell t/T} - x^*(t) \right] dt$$

$$P_e = \frac{1}{T} \int_{-T/2}^{T/2} \Bigg\{$$

$$\left[\frac{1}{T} \sum_{k=-N+1}^{N-1} X_k \, e^{j2\pi kt/T}\right] \left[\frac{1}{T} \sum_{\ell=-N+1}^{N-1} X_\ell{}^* \, e^{-j2\pi \ell t/T}\right]$$

$$- \left[\frac{1}{T} \sum_{k=-N+1}^{N-1} X_k \, e^{j2\pi kt/T}\right] x^*(t) -$$

$$x(t) \left[\frac{1}{T} \sum_{\ell=-N+1} X_\ell^* \, e^{-j2\pi \ell t/T}\right] + x(t)x^*(t) \Bigg\} \, dt$$

$$P_e = \frac{1}{T^3} \sum_{k=-N+1}^{N-1} \sum_{\ell=-N+1}^{N-1} X_k X_\ell{}^* \int_{-T/2}^{T/2} e^{j2\pi(k-\ell)t/T}\, dt$$

$$- \frac{1}{T^2} \sum_{k=-N+1}^{N-1} X_k \int_{-T/2}^{T/2} x^*(t)\, e^{j2\pi kt/T} dt$$

$$- \frac{1}{T^2} \sum_{\ell=-N+1}^{N-1} X_\ell^* \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi\ell t/T}\, dt$$

$$+ \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

$$\int_{-T/2}^{T/2} e^{j2\pi(k-\ell)t/T}\, dt = \frac{T}{j2\pi(k-\ell)}\, e^{j2\pi(k-\ell)t/T} \Bigg|_{-T/2}^{T/2}$$

$$= \frac{T}{j2\pi(k-\ell)} \left[e^{j\pi(k-\ell)} - e^{-j\pi(k-\ell)}\right]$$

$$= T\,\mathrm{sinc}\,(k-\ell)$$

$$= \begin{cases} T, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

Let $\tilde{X}_\ell \triangleq \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi\ell t/T}\, dt$

$$P_e = \frac{1}{T^2} \sum_{k=-N+1}^{N-1} \{ | X_k |^2 - X_k \tilde{X}_k^* - X_k^* \tilde{X}_k \} + P_x$$

X_k - unknown coefficients in the Fourier series approximation

$\tilde{X}_k$ - fixed numbers that depend on x(t)

We want to choose values for the coefficients X_k, $k = -N+1, \ldots, N-1$ which will minimize P_e.

Fix ℓ between $-N+1$ and $N-1$.

Let $X_\ell = A_\ell + j\,B_\ell$,

and consider $\dfrac{\partial P_e}{\partial A_\ell}$ and $\dfrac{\partial P_e}{\partial B_\ell}$.

$$P_e = \frac{1}{T^2} \sum_{k=-N+1}^{N-1} \{ | X_k |^2 - X_k \tilde{X}_k^* - X_k^* \tilde{X}_k \} + P_x$$

$$\frac{\partial P_e}{\partial A_\ell} = \frac{1}{T^2} \frac{\partial P_e}{\partial A_\ell} \Bigg\{ (A_\ell + j B_\ell)(A_\ell - j B_\ell)$$

$$- (A_\ell + j B_\ell) \tilde{X}_\ell^* - (A_\ell - j B_\ell) \tilde{X}_\ell \Bigg\}$$

$$= \frac{1}{T^2} \Bigg\{ X_\ell^* + X_\ell - \tilde{X}_\ell^* - \tilde{X}_\ell \Bigg\}$$

$$= \frac{2}{T^2} \, \mathrm{Re}\{X_\ell - \tilde{X}_\ell\}$$

$$\frac{\partial P_e}{\partial A_\ell} = 0 \quad \Rightarrow \quad \mathrm{Re}\{X_\ell\} = \mathrm{Re}\{\tilde{X}_\ell\}$$

Similarly

$$\frac{\partial P_e}{\partial B_\ell} = \frac{1}{T^2}\,\{jX_\ell^* - jX_\ell - j\tilde{X}_\ell^* + j\tilde{X}_\ell\}$$

$$= j\,\frac{2}{T^2}\,\mathrm{Im}\,\{\tilde{X}_\ell - X_\ell\}$$

and

$$\frac{\partial P_e}{\partial B_\ell} = 0 \quad \Rightarrow \quad \mathrm{Im}\,\{X_\ell\} = \mathrm{Im}\{\tilde{X}_\ell\}$$

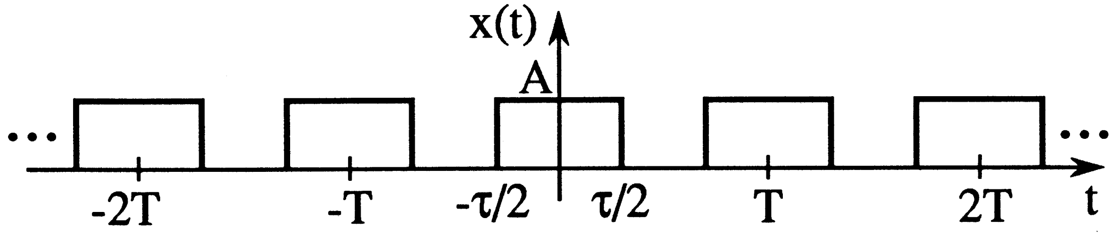
To summarize:

For $$\hat{x}(t) = \frac{1}{T} \sum_{k=-N+1}^{N-1} X_k \, e^{j2\pi kt/T}$$

to be a minimum mean-squared error approximation to the signal x(t) over the interval $-T/2 \leq t \leq T/2$, the coefficients X_k must satisfy

$$X_k = \int_{-T/2}^{T/2} x(t) \, e^{-j2\pi kt/T} \, dt$$

Example



$$X_k = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T}\, dt$$

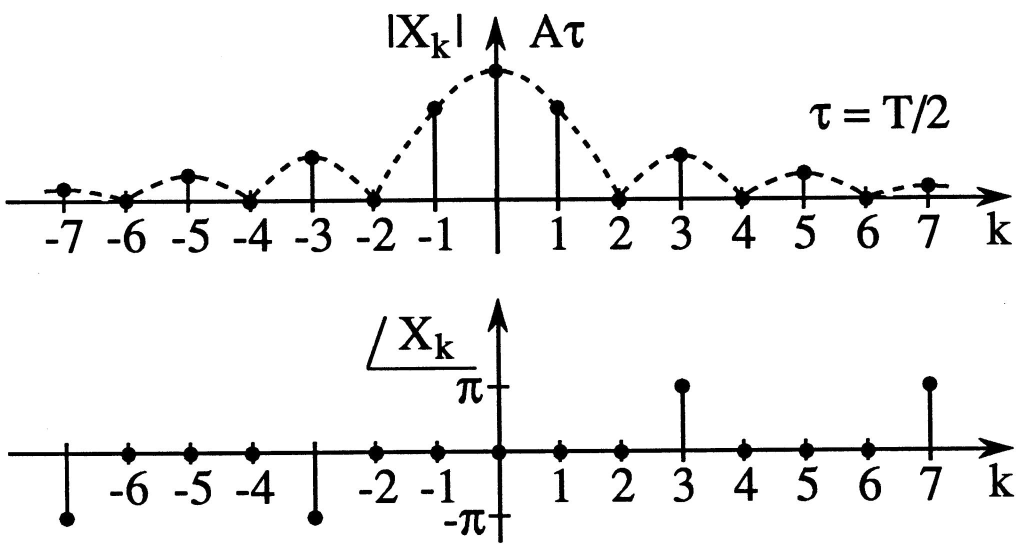
$$= A \int_{-\tau/2}^{\tau/2} e^{-j2\pi kt/T}\, dt = \frac{A}{-j2\pi k/T}\, e^{-j2\pi kt/T} \Bigg|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j2\pi k/T} \left[e^{-j\pi k\tau/T} - e^{j\pi k\tau/T} \right] = A\tau\, \frac{\sin(\pi k\tau/T)}{\pi k\tau/T}$$

Line Spectrum

$$X_k = A\tau \,\text{sinc}\,(k\tau/T)$$

$$|\,X_k\,| = A\tau \;|\,\text{sinc}\,(k\tau/T)\,|\,, \quad \underline{/X_k} = \begin{cases} 0, & \text{sinc}\,(k\tau/T) > 0 \\ \pm\,\pi, & \text{sinc}\,(k\tau/T) < 0 \end{cases}$$

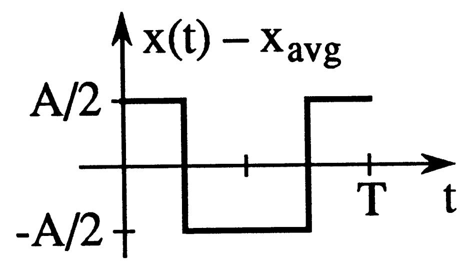


Comments

Recall $X_k = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T}\, dt$

1. $X_0 = \int_{-T/2}^{T/2} x(t)\, dt = T\, x_{avg}$

2. $X_k = 0, \;\; k \neq 0$ and even because of odd half wave symmetry



3. $\lim_{k \to \infty} |X_k| = 0$

What happens as N increases?

Let $\hat{x}_N(t) = \frac{1}{T} \sum_{k=-N+1}^{N-1} X_k \, e^{j2\pi kt/T}$

1. If $\int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$, $\int_{-T/2}^{T/2} |\hat{x}_N(t) - x(t)|^2 dt \rightarrow 0$

2. If $\int_{-T/2}^{T/2} |x(t)| \, dt < \infty$ and other mild (Dirichlet) conditions are met, $\hat{x}_N(t) \rightarrow x(t)$ for all t where x(t) is continuous.

3. In the neighborhood of discontinuities, $\hat{x}_N(t)$ exhibits an overshoot or undershoot with maximum amplitude equal to 9% of the step size no matter how large N is (Gibbs phenomena).