APPENDIX B

THE EXPECTED VALUE OF THE MOMENTS

Case 1: $E[M_0]$

$$E[M_0] = T^{-1} \int_t \int_\tau E[\xi(t)\xi^*(\tau)]G_T(t)G_T(\tau)\delta(\tau - t)d\tau dt$$

$$= T^{-1} \int_t \int_\tau R_\xi(t - \tau)G_T(t)G_T(\tau)\delta(\tau - t)d\tau dt$$

$$= T^{-1} \int_t R_\xi(0)G_T(t)dt. \tag{B.1}$$

Therefore,

$$E[M_0] = R_\xi(0).$$

But,

$$R(\tau) = \int_f S(f)e^{j2\pi f\tau}df. \tag{B.2}$$

Hence,

$$R_\xi(0) = \int_f S_\xi(f)df.$$

Therefore,

$$E[M_0] = P \tag{B.3}$$

where P is the total power in the spectrum.

Case 2: $E[M_1]$

$$E[M_1] = (j2\pi T)^{-1} \int_t \int_\tau R_\xi(t - \tau)G_T(t)G_T(\tau)\dot{\delta}(\tau - t)d\tau dt$$

$$= (j2\pi T)^{-1} \int_t G_T(t)[\dot{R}_\xi(0)G_T(t) - R_\xi(0)\dot{G}_T(t)]dt. \tag{B.4}$$

Integrating yields

$$E[M_1] = (j2\pi)^{-1}\left\{\dot{R}_\xi(0) - R_\xi(0)\left[\frac{1}{2} - \frac{1}{2}\right]\right\}. \quad (B.5)$$

But,

$$\dot{R}_\xi(0) = 2\pi j \int_f fS_\xi(f)df. \quad (B.6)$$

Therefore,

$$E[M_1] = \int_f fS_\xi(f)df \quad (B.7)$$

and since $\rho(t)$ is a real function,

$$E[M_1] = f_a P. \quad (B.8)$$

Case 3: $E[M_2 - \hat{P}B_T^2]$

From Chapter II,

$$\hat{P}B_T^2 = -(4\pi^2 T)^{-1} \int_t G_T(t)A^2(t)\ddot{G}(t)dt$$

where $A^2(t) = \xi(t)\xi^*(t)$. Therefore,

$$M_2 - \hat{P}B_T^2 = -(4\pi^2 T)^{-1} \int_t \{G_T(t) \int_\tau \xi(t)\xi^*(\tau)G_T(\tau)\delta(\tau - t)d\tau - \xi(t)\xi^*(t)\ddot{G}_T(t)\}dt \quad (B.9)$$

and

$$E[M_2 - \hat{P}B_T^2] = -(4\pi^2 T)^{-1} \cdot \int_t G_T(t)\{\int_\tau R_\xi(t - \tau)G_T(\tau) \cdot \ddot{\delta}(\tau - t)d\tau - R_\xi(0)\ddot{G}_T(t)\}dt. \quad (B.10)$$

Integrating with respect to τ yields

$$E[M_2 - \hat{P}B_T^2] = -(4\pi^2 T)^{-1}$$

$$\cdot \int_t G_T(t)\{\ddot{R}_\xi(0) - 2\dot{R}_\xi(0)\dot{G}_T(t) \qquad (B.11)$$

$$+ R_\xi(0)\ddot{G}_T(t) - R_\xi(0)\ddot{G}_T(t)\}dt.$$

But,

$$\ddot{R}_\xi(0) = -4\pi^2 \int f^2 S_\xi(f)df$$

$$= -4\pi^2[B^2 + B_\rho^2 + f_a^2]P. \qquad (B.12)$$

Therefore,

$$E[M_2 - \hat{P}B_T^2] = -(4\pi^2 T)^{-1}\{- 4\pi^2 TP[B^2 + B_\rho^2 + f_a^2]$$

$$- 4\pi j f_a P \frac{1}{2} - \frac{1}{2} + 0\}$$

$$= P[B^2 + B_\rho^2 + f_a]. \qquad (B.13)$$